Lecture 18: Infinitesimal Strain

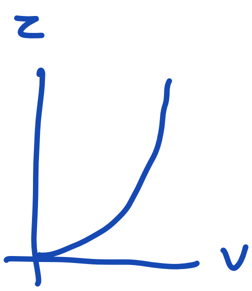
Logistics:
- HW5 all submitted → returned soon
- HW6 still missing 4 submissions
- HW7 due next week

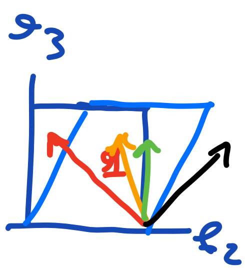
Last time: Cauchy-Green Strain Relations

stretch: $\lambda(\hat{\underline{Y}}) = \sqrt{\hat{\underline{Y}} \cdot \underline{\underline{C}}\, \hat{\underline{Y}}} > 0$

shear: $\gamma(\hat{\underline{Y}}, \hat{\underline{Z}}) = \Theta(\hat{\underline{Y}}, \hat{\underline{Z}}) - \theta(\hat{\underline{Y}}, \hat{\underline{Z}})$

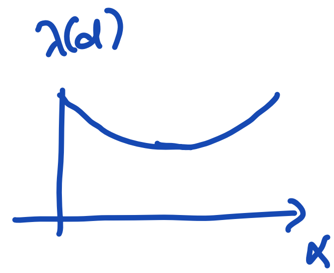
$$\theta(\hat{\underline{Y}}, \hat{\underline{Z}}) = \frac{\hat{\underline{Y}} \cdot \underline{\underline{C}}\, \hat{\underline{Z}}}{\sqrt{\hat{\underline{Y}} \cdot \underline{\underline{C}}\, \hat{\underline{Y}}}\ \sqrt{\hat{\underline{Z}} \cdot \underline{\underline{C}}\, \hat{\underline{Z}}}}$$



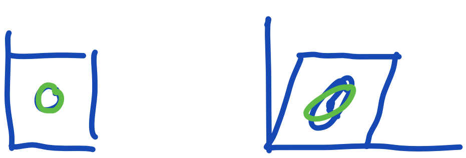


$\lambda(\underline{d})$

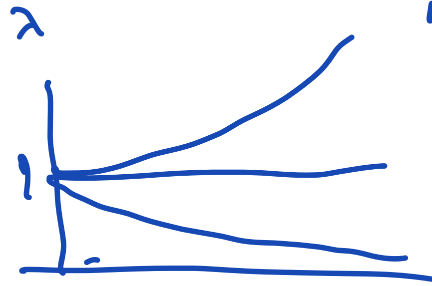
$\underline{d} = \frac{1}{\sqrt{2}}(\underline{e}_3 - \underline{e}_2)$



$\lambda^2(\underline{d}) = \underline{d} \cdot \underline{\underline{C}}\, \underline{d}$



α



Finite Strain

Today: Infinitesimal Strain

Infinitesimal Strain Tensor

⇒ linear elasticity

displacement: $\underline{u} = \underline{x} - \underline{X} = \underline{\varphi}(\underline{X}) - \underline{X}$

$$\underline{\underline{\varepsilon}} = \text{sym}(\nabla \underline{u}) = \frac{1}{2}(\nabla \underline{u} + \nabla \underline{u}^T)$$

linear in $\underline{u}$

How is $\underline{\underline{\varepsilon}}$ related $\underline{\underline{F}}$ and $\underline{\underline{C}}$?

$$\nabla \underline{u} = \nabla(\underline{\varphi} - \underline{X}) = \nabla \underline{\varphi} - \underline{\underline{I}} = \underline{\underline{F}} - \underline{\underline{I}}$$

$$\underline{\underline{\varepsilon}} = \text{sym}(\underline{\underline{F}} - \underline{\underline{I}}) = \frac{1}{2}\left[(\underline{\underline{F}} - \underline{\underline{I}})^T + (\underline{\underline{F}} - \underline{\underline{I}})\right] =$$

$$= \frac{1}{2}\left[\underline{\underline{F}}^T - \underline{\underline{I}} + \underline{\underline{F}} - \underline{\underline{I}}\right]$$

$$= \frac{1}{2}(\underline{\underline{F}} + \underline{\underline{F}}^T) - \underline{\underline{I}}$$

$$\underline{\underline{\varepsilon}} = \frac{1}{2}(\underline{\underline{F}} + \underline{\underline{F}}^T) - \underline{\underline{I}}$$

$$\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}}$$

$$\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}})$$

Euler Lagrange

$$\underline{\underline{F}} = \nabla \underline{u} + \underline{\underline{I}}$$

$$\underline{\underline{C}} = (\nabla \underline{u} + \underline{\underline{I}})^T (\nabla \underline{u} + \underline{\underline{I}})$$

$$= \nabla u^T \nabla u + \underbrace{\nabla u + \nabla u^T}_{2\underline{\underline{\varepsilon}}} + \underline{\underline{I}}$$

$$\underline{\underline{\varepsilon}} = \underbrace{\frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}})}_{\underline{\underline{E}}} - \frac{1}{2} \nabla u^T \nabla u = \underline{\underline{E}} - \frac{1}{2} \nabla u^T \nabla u$$

We say φ is small if $|\nabla u| \ll 1$

$|\nabla u| \to 0 \qquad \underline{\underline{\varepsilon}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}}) \approx \underline{\underline{E}}$

$\underline{\underline{\varepsilon}}$ is linear in $\nabla u, u$

$\underline{\underline{E}}, \underline{\underline{C}}$ is non linear in ∇u and u

Components of $\underline{\underline{\varepsilon}}$

$$\varepsilon_{ii} \approx \lambda(\underline{e}_i) - 1$$

$$\varepsilon_{ij} \approx \frac{1}{2} \sin(\gamma(\underline{e}_i, \underline{e}_j)) \approx \frac{1}{2} \gamma(\underline{e}_i, \underline{e}_j) = \tau$$

Diagonal:

$$\underline{\underline{C}} = 2\underline{\underline{\varepsilon}} + I + \nabla \underline{u}^T \nabla \underline{u}$$

$$C_{ii} = 2\varepsilon_{ii} + 1 + \text{hot}$$

$$\lambda(\underline{e}_i) = \sqrt{C_{ii}} = \sqrt{2\varepsilon_{ii} + 1}$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \dots \quad \text{(Taylor Series)}$$

$$\sqrt{C_{ii}} = 1 + \varepsilon_{ii} + \text{hot}$$

$$\Rightarrow \quad \varepsilon_{ii} = \sqrt{C_{ii}} - 1 = \lambda(\underline{e}_i) - 1$$

$$\lambda(\underline{e}_i) = \frac{|\underline{y} - \underline{x}|}{|\underline{Y} - \underline{X}|} = \frac{\ell}{L}$$

dl: ℓ ~ L

$$\varepsilon_{ii} = \lambda(\underline{e}_i) - 1 = \frac{|\underline{y} - \underline{x}|}{|\underline{Y} - \underline{X}|} - 1 = \frac{|\underline{y} - \underline{x}| - |\underline{Y} - \underline{X}|}{|\underline{Y} - \underline{X}|}$$

L

$$\varepsilon_{ii} = \frac{dL}{L} \quad \text{relative change in length}$$

Off-Diagonal components:

$$\sin(\gamma(\underline{e}_i, \underline{e}_j)) = \frac{C_{ij}}{\sqrt{C_{ii}}\sqrt{C_{jj}}}$$

$\delta \ll 1$

$|\nabla u| = O(\delta)$

$$C_{ij} = 2\varepsilon_{ij} + O(\delta^2) \qquad i \neq j$$

$$C_{ii} = 1 + O(\delta)$$

$$\sqrt{C_{ii}}\sqrt{C_{jj}} = (1 + O(\delta))(1 + O(\delta)) = 1 + O(\delta^2)$$

$$1 + \underbrace{2\,O(\delta)}_{?} + O(\delta^2)$$

$$\sin(\gamma(\underline{e}_i, \underline{e}_j)) = C_{ij} = 2\,\varepsilon_{ij} + \text{hot}$$

$$\Rightarrow \quad \varepsilon_{ij} \approx \frac{1}{2}\sin(\gamma(\underline{e}_i, \underline{e}_j))$$

Green-Lagrange Strain Tensor

$$\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}}) \qquad |\nabla \underline{u}| \ll 1$$

$$\underline{\underline{E}} \rightarrow \underline{\underline{\varepsilon}}$$

Linearization of Kinematic Quantities

$$\underline{\underline{F}} = \nabla \underline{\varphi}$$

$$\underline{\underline{H}} = \nabla \underline{u} = \underline{\underline{F}} - \underline{\underline{I}}$$

what is linearization of $\underline{\underline{U}}, \underline{\underline{V}}, \underline{\underline{R}}, \underline{\underline{C}}, \underline{\underline{E}}$

in limit $|\underline{\underline{H}}| \ll 1$

$$|\underline{\underline{H}}| = \sqrt{\underline{\underline{H}} : \underline{\underline{H}}} = \sqrt{H_{ij} H_{ij}} = \sqrt{H_{11}^2 + H_{12}^2 \ldots + H_{33}^2} = \delta$$

$\underline{\underline{Z}}(\underline{\underline{H}})$ tensor-valued tensor function

$\underline{\underline{Z}}(\underline{\underline{H}}) = O(|H|^n)$ if

$$|\underline{\underline{Z}}(\underline{\underline{H}})| \le \alpha |\underline{\underline{H}}|^n \quad \text{as} \quad |\underline{\underline{H}}| \to 0$$

For any sym. $\underline{\underline{A}}$ and $m \in \mathbb{R}$

$$(\underline{\underline{I}} + \underline{\underline{A}})^m = \underline{\underline{I}} + m\underline{\underline{A}} + O(|A|^2)$$

Linearization of $\underline{\underline{R}}$:

$$\underline{\underline{C}} = \underline{\underline{U}}^2 = \underline{\underline{F}}^T\underline{\underline{F}} = \underline{\underline{I}} + \underline{\underline{H}} + \underline{\underline{H}}^T + O(|H|^2)$$

$$\underline{\underline{R}} = \underline{\underline{F}}\,\underline{\underline{U}}^{-1} = (\underline{\underline{H}} + \underline{\underline{I}})(\underline{\underline{I}} + \underline{\underline{\varepsilon}})^{-1}$$

$$= (H + I)(\underline{\underline{I}} - \underline{\underline{\varepsilon}} + O(|H|^2))$$

$$= I + \underline{\underline{H}} - \tfrac{1}{2}(\underline{\underline{H}} + \underline{\underline{H}}^T)$$

$$\underline{\underline{R}} \simeq \underline{\underline{I}} + \tfrac{1}{2}(H - H^T)$$

$$\boxed{\underline{\underline{R}} = \underline{\underline{I}} + \tfrac{1}{2}(\nabla u - \nabla u^T)}$$

$$\underline{\underline{\omega}} = \text{skew}(\nabla \underline{u})$$

finite strain: (non. linear)

$$\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} \qquad \text{multiplicative}$$

infinitesimal (linearized)

$$\underline{\underline{F}} = \nabla \underline{u} + \underline{\underline{I}} = \underline{\underline{I}} + \text{sym}(\nabla \underline{u}) + \text{skew}(\nabla \underline{u})$$

$$\boxed{\underline{\underline{F}} = \underline{\underline{I}} + \underline{\underline{\varepsilon}} + \underline{\underline{\omega}}}$$

$\underline{\underline{\varepsilon}}$ ↑ stretch "$\underline{\underline{U}}$"

$\underline{\underline{\omega}}$ ↑ rotation "$\underline{\underline{R}}$"

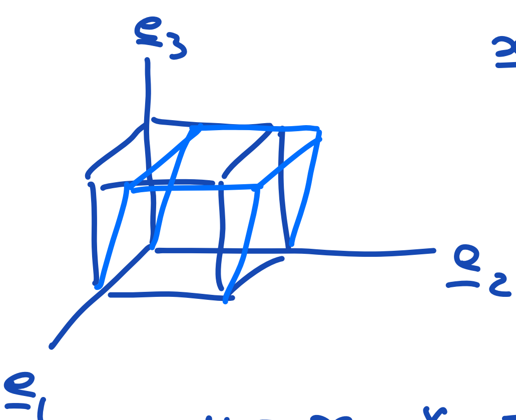
$$\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} = \left(\underline{\underline{I}} + \underline{\underline{\omega}} + O(\delta^2)\right)\left(\underline{\underline{I}} + \underline{\underline{\varepsilon}} + O(\delta^2)\right)$$

$$= \underline{\underline{I}} + \underline{\underline{\omega}} + \underline{\underline{\varepsilon}} + \cancel{\underline{\underline{\omega}}\,\underline{\underline{\varepsilon}}}\ O(\delta^2)$$

Finite strain stretch and rotation are multiplicative

Infinitesimal strain they are additive.

Try Example:



$$\underline{x} = \varphi(\underline{X}) = \begin{bmatrix} X_1 \\ X_2 + \alpha X_3 \\ X_3 \end{bmatrix}$$

$$F = \nabla\varphi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{u} = \underline{x} - \underline{X} = \begin{bmatrix} 0 \\ \alpha X_3 \\ 0 \end{bmatrix}$$

$$\underline{\underline{C}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & \alpha & 1+\alpha^2 \end{bmatrix}$$

$$\nabla \underline{u} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\nabla u + \nabla u^{T}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha/2 \\ 0 & \frac{\alpha}{2} & 0 \end{bmatrix}$$

Streches:

$$\varepsilon_{ii} = \lambda(\underline{e}_i) - 1$$

$$\varepsilon_{11} = \lambda(\underline{e}_i) - 1 = 0 \quad \Rightarrow \quad \lambda(\underline{e}_i) = 1$$

$\Rightarrow$ no strech in any coord.

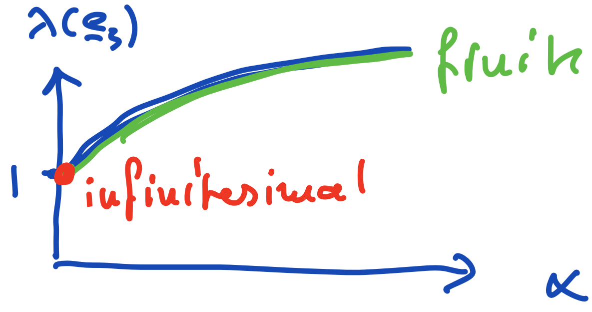
$$C_{ii} = \lambda^2(\underline{e}_i)$$

$\lambda(\underline{e}_1) = \lambda(\underline{e}_1) = 1$ $\Rightarrow$ no strech

$$C_{33} = \lambda^2(\underline{e}_3) = 1 + \alpha^2$$

$$\lambda(\underline{e}_3) = \sqrt{1+\alpha^2}$$

$$\lim_{\alpha \to 0} \lambda(\underline{e}_3) = 0$$ no strech



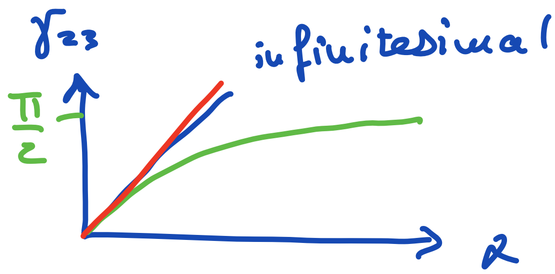
Shear:

$$e_{ij} \approx \frac{1}{2} \sin(\gamma(\underline{e}_i, \underline{e}_j))$$

$\gamma \ll 1$

$$\gamma(\underline{e}_i, \underline{e}_j) = 2\, \varepsilon_{ij}$$

$$\gamma(\underline{e}_2, \underline{e}_3) = 2\, \varepsilon_{23} = \alpha$$



finite:

$$\gamma(\underline{e}_2, \underline{e}_3) = \operatorname{asin}\left(\frac{\alpha}{\sqrt{1+\alpha^2}}\right)$$