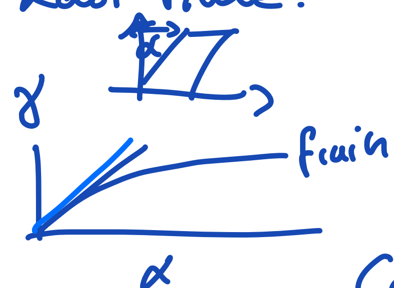
Lecture 19: Motion and Material Derivative

Logistics:
- HW 7 due Th
- HW 6 → please complete
- HW 5 → graded

Last time: - Infinitesimal strain tensor



$$\underline{\underline{\varepsilon}} = \tfrac{1}{2}\left(\nabla \underline{u} - \nabla \underline{u}^T\right) \qquad \underline{u} = \underline{\varphi} - \underline{X}$$

$$\lim_{|\nabla u| \to 0} \underline{\underline{E}} = \underline{\underline{\varepsilon}} \qquad \underline{\underline{E}} = \tfrac{1}{2}\left(\underline{\underline{C}} - \underline{\underline{I}}\right)$$

Components: $\varepsilon_{ii} \approx \lambda(\underline{e}_i) - 1$

$\varepsilon_{ij} \approx \tfrac{1}{2} \sin \gamma(\underline{e}_i, \underline{e}_j)$

- Linearization:

$$\underline{\underline{C}} \approx \underline{\underline{I}} + \nabla \underline{u} + \nabla \underline{u}^T \qquad \text{sym}(\nabla u)$$

$$\underline{\underline{U}} \approx \underline{\underline{I}} + \tfrac{1}{2}\left(\nabla \underline{u} + \nabla u^T\right) = \underline{\underline{I}} + \underline{\underline{\varepsilon}}$$

$$\underline{\underline{R}} \approx \underline{\underline{I}} + \tfrac{1}{2}\left(\nabla \underline{u} - \nabla u^T\right) = \underline{\underline{I}} - \underline{\underline{\omega}}$$

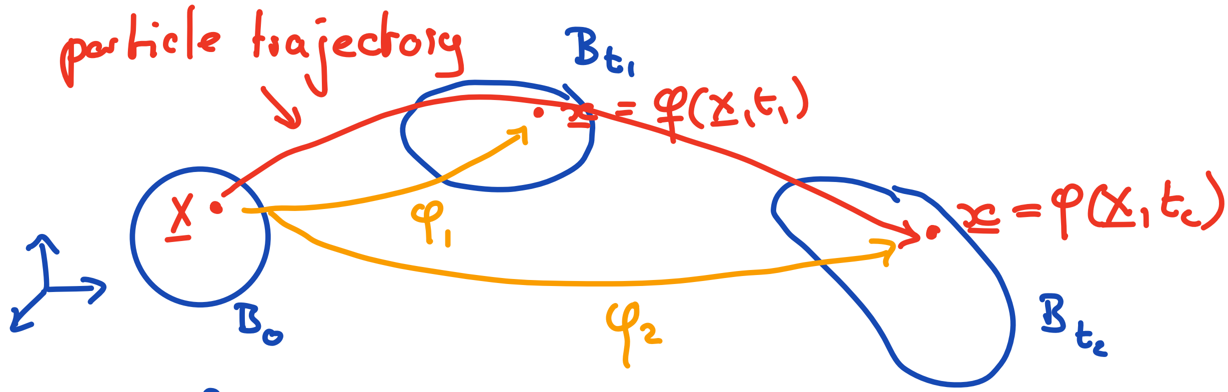
skew(∇u)

$$\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} \approx \underline{\underline{I}} + \underline{\underline{\varepsilon}} + \underline{\underline{\omega}}$$

Today: Motion → $\varphi(\underline{X}, \underline{t})$

Material time derivative

Motion



B_0: • undeformed
- reference
- material
- initial

B_t: • deformed
- spatial
- current

Continuous deformation of body over time is a motion

$$\boxed{\underline{x}(t) = \varphi_t(\underline{X}) = \varphi(\underline{X}, t)}$$

assume φ_t smooth $\Rightarrow$ admissible

$\Rightarrow$ Inverse motion $\boxed{\underline{X} = \underline{\psi}_t(\underline{x}) = \underline{\psi}(\underline{x}, t)}$

Material and spatial fields

Temperature is naturally a spatial field: $T(\underline{x}, t)$

Velocity is naturally a material field: $V(\underline{X}, t)$

φ_t & ψ_t : material $\leftrightarrow$ spatial

Material field: $\Omega = \Omega(\underline{X}, t)$ $\qquad X = \psi_t(x)$

Spatial field: $\Gamma = \Gamma(\underline{x}, t)$ $\qquad \underline{x} = \varphi(\underline{X}, t)$

Spatial description of material field

$$\Omega_s(\underline{x}, t) = \Omega(\psi(\underline{x}, t), t)$$

Material description of a spatial field

$$\Gamma_m(\underline{X}, t) = \Gamma(\varphi(\underline{X}, t), t)$$

Coordinate derivatives:

Material coordinates: ∇_X = Grad, Div, Curl

Spatial coordinates: ∇_x

Velocity and Acceleration fields

The velocity and acceleration of a material particle $\underline{X}$ at time t due to the motion $\varphi(\underline{X},t)$ are:

$$\boxed{\begin{aligned} V(\underline{X},t) &= \frac{\partial}{\partial t}\underbrace{\varphi(X,t)}_{\underline{x}} = \frac{\partial x}{\partial t}\Big|_{\underline{X}} \\ A(\underline{X},t) &= \frac{\partial^2}{\partial t^2}\varphi(\underline{X},t) = \frac{\partial^2 x}{\partial t^2}\Big|_{\underline{X}} \end{aligned}}$$

naturally material fields

spatial descriptions:

$$\underline{v}(\underline{x},t) = V_s(\underline{x},t) = \frac{\partial}{\partial t}\varphi(\psi(\underline{x},t),t)$$

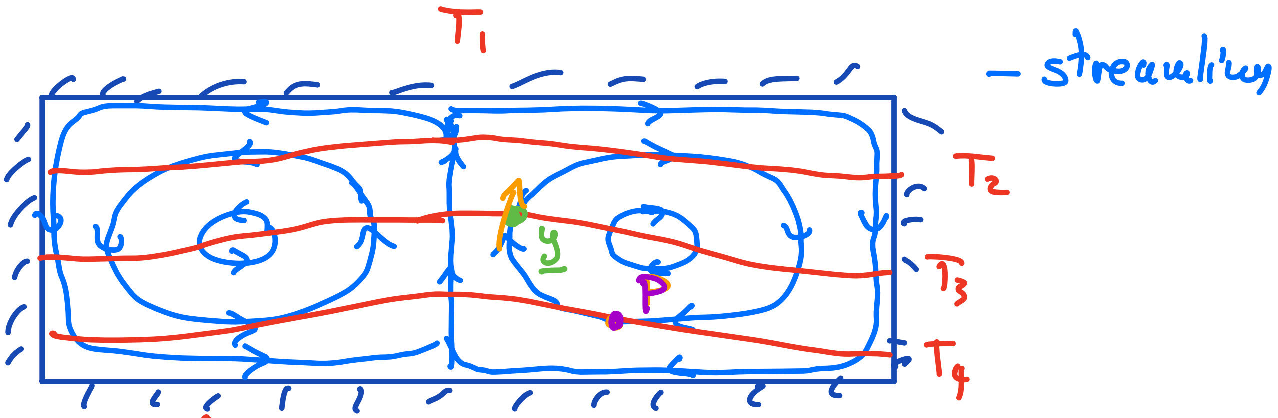
$$\underline{a}(\underline{x},t) = A_s(\underline{x},t) = \frac{\partial^2}{\partial t^2}\varphi(\psi(\underline{x},t),t)$$

The spatial fields $\underline{v}$ and $\underline{a}$ correspond to the material particle that passes through $\underline{x}$ at t.

Note: $\underline{a} \neq \frac{\partial v}{\partial t}$

Example: Steady Convection

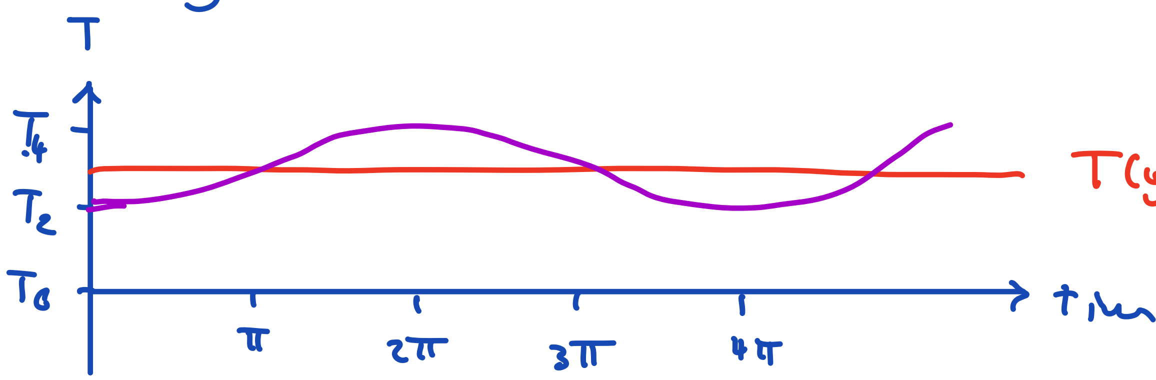
$T_4 > T_1$

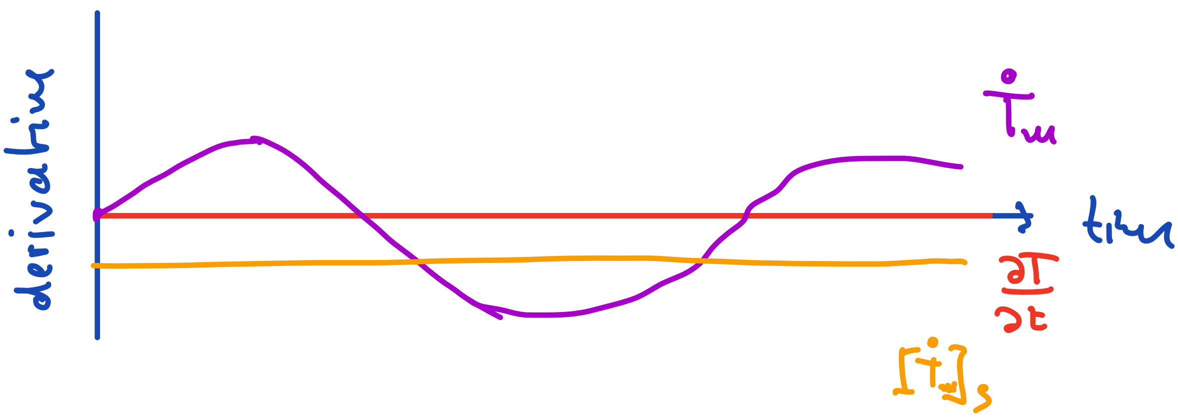


$\dot{T}(x,t) = \cancel{\frac{\partial T}{\partial t}}^{0} + \underline{\nabla T} \cdot \underline{v}$

T_4

Steady state: $T(\underline{x}, t) = T(\underline{x})$ (spatial field)





1) $\cdot\ T(y)$ $\quad \frac{\partial T}{\partial t} = 0$

2) Particle P initially at $\underline{X}$ now at $\underline{x} = \varphi(\underline{X}, t)$ passes through y every 2π

$T(\varphi(\underline{X}, t)) = T_m(X, t)$ oscillates between T_2 & T_4

Its derivative "material derivative"

$\dot{T}_m(\underline{X}, t) \neq 0$

3) Consider particles passing through y with initial locations $\underline{Y} = \psi(y, t)$.

What is change in temperature these particles experience as they pass through y ? $\dot{T}_m(\psi(y, t)) = [\dot{T}_m]_s$

spatial representation of material time derivative

$$x = \underline{X} t + \exp(-\lambda t)$$

$$\dot{x} = \underline{X} \cancel{+} \lambda \exp(-\lambda t)$$

Different time derivatives

I) Material time derivative of a material field

$$\boxed{\dot{\Omega}(\underline{X},t) = \frac{D\Omega}{Dt}(\underline{X},t) = \frac{\partial \Omega}{\partial t}\Big|_{\underline{X}}}$$

derivative of Ω with respect to t holding X fixed

Called: total, substantial, convective material derivative

$\dot{\Omega}$ represents rate of change of Ω seen by an observer following the particle.

II Spatial time derivative of a spatial field

$$\boxed{\frac{\partial \Gamma}{\partial t}(\underline{x},t)\Big|_{\underline{x}} = \frac{\partial \Gamma}{\partial t}}$$

local time derivative

Rate of change of Γ seen by stationary observer at $\underline{x}$

III Material time derivative of spatial field

Derivative of scalar field Γ with respect to time t, holding $\underline{X}$ fixed.

$$\Rightarrow \qquad \underline{x} = \varphi(\underline{X}, t)$$

$$\boxed{\dot{\Gamma}(\underline{x},t) = \frac{\partial}{\partial t}\, \underbrace{\Gamma(\varphi(\underline{X},t),t)}_{\Gamma_m(\underline{X},t)} \Big|_{\underline{X}=\psi(\underline{x},t)}}$$

$\Rightarrow$ two time dependencies, one explicit, other through the motion $\varphi(\underline{X},t)$

By chain rule:

$$\frac{\partial}{\partial t}\Gamma(\underset{\underline{x}=\varphi(\underline{X},t)}{\underbrace{\varphi(\underline{X},t)}},t) = \frac{\partial \Gamma}{\partial t}\Big|_{\underline{x}=\varphi(\underline{X},t)} + \frac{\partial \Gamma}{\partial x_i}(\underline{x},t)\Big|_{\underline{x}=\varphi(\underline{X},t)} \underset{\frac{\partial \varphi_i}{\partial t}(\underline{X},t)}{\underbrace{\frac{\partial \varphi_i}{\partial t}(x_i}}$$

recognize spatial velocity: $v_i(\underline{x},t)\big|_{\underline{x}=\varphi(\underline{X},t)} = \frac{\partial \varphi_i}{\partial t}$

substitution

$$\dot{\Gamma}(\underline{X},t) = \left[\frac{\partial \Gamma}{\partial t}(\underline{x},t) + \frac{\partial \Gamma}{\partial x_i}(\underline{x},t)\; v_i(\underline{x},t)\right]_{\underline{x}=\varphi(\underline{X},t)}$$

Expressing this in spatial coordinates

$$\dot{\Gamma}(\underline{x},t) = \frac{\partial \Gamma}{\partial t} + \frac{\partial \Gamma}{\partial x_i} v_i$$

Let $\varphi(\underline{X},t)$ be motion with spatial velocity field $\underline{v}$ and scalar spatial field $\phi(\underline{x},t)$ vector spatial field $\underline{w}$ the the spatial representations of the material time derivative are:

$$\boxed{\begin{aligned} \dot{\phi} &= \frac{\partial \phi}{\partial t} + \nabla_x \phi \cdot \underline{v} \\ \underline{\dot{w}} &= \frac{\partial \underline{w}}{\partial t} + (\nabla_x \underline{w})\underline{v} \end{aligned}}$$ ← familiar

Fluid mechanics: $(\nabla_{\underline{x}} \underline{w})\underline{v} = \underline{v} \cdot \nabla \underline{w}$

These results are important because they allow the computation of $\dot{\phi}$ and $\underline{\dot{w}}$ without knowledge of φ if $\underline{v}$ is know!

⇒ in fluid mechanics we never see φ

The spatial acceleration can be computed as

$$\boxed{\underline{a}(\underline{x},t) = \dot{\underline{v}}(\underline{x},t) = \frac{\partial \underline{v}}{\partial t} + \underbrace{(\nabla_{\underline{x}} \underline{v})\,\underline{v}}_{\text{non-linear}}}$$

$\Rightarrow$ basic non-linearity in
Fluid mechanics (Navier-Stokes)