Lecture 2: Mass & Force, Isostacy

Logistics: → please fill out office hours poll
- video from last lecture o.k.
- post HW1 today due next Th
- piazza working ? ✓

Last time:
- intro
- vector review
- index notation

Dummy indices : $\underline{a} = \sum a_i \underline{e}_i$

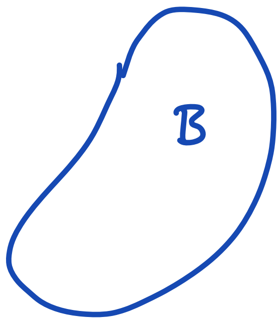
Free indices : $c_j = a_i b_i a_j$

Kronecker delta : δ_{ij} → dot product

Today:
- Mass & density
- Forces & Torques
- Weight & Buoyancy

⇒ Hydrostatic eqbm
- Geo application is Isostacy
- Finish index notation

Continuum Mass & Force concepts



Volume of $\mathcal{B}$:

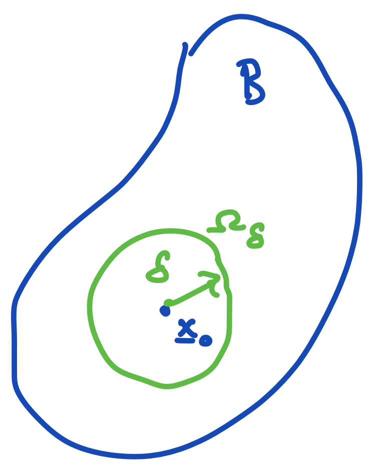
$$V_B = \int_B dV$$

Mass of $\mathcal{B}$: $m_B = \int \rho(\underline{x})\, dV$

$\rho(\underline{x})$ = mass density field

At any point $\underline{x}_0$ in $\mathcal{B}$

$$\boxed{\rho(\underline{x}_0) = \lim_{\delta \to 0} \frac{m_{\Omega_\delta}}{V_{\Omega_\delta}}}$$



Important geom. quantities:

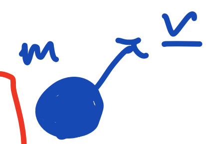
Center of Volume: $\underline{x}_V = \frac{1}{V_B} \int_B \underline{x}\, dV$

Center of mass: $\underline{x}_m = \frac{1}{m_B} \int_B \rho(\underline{x})\, \underline{x}\, dV$

$\Rightarrow$ resulting forces

Short review of force & momentum

Object with mass m and velocity $\underline{v}$ has momentum:



Linear momentum: $\boxed{\underline{L} = m\,\underline{v}}$

angular momentum: $\boxed{\underline{j} = (\underline{x} - \underline{z}) \times \underline{L}}$

always relative to point $\underline{z}$

Newtons 1st law: "Principle of inertia"

fixed frame of reference object preserves motion unless acted upon by force or torque

Force: $\underline{f} = \frac{d\underline{L}}{dt} = \frac{d(m\underline{v})}{dt} = m \underbrace{\frac{d\underline{v}}{dt}} = m\,\underline{a}$

$\left[\frac{ML}{T^2}\right] = N$

$\dot{\underline{v}} = \underline{a} = \ddot{\underline{x}}$

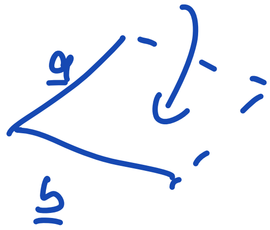
$\boxed{\underline{f} = m\,\underline{a}}$ 2nd law

Torque: $\underline{\tau} = \frac{d\underline{j}}{dt} = \frac{d}{dt}\left[(\underline{x} - \underline{z}) \times (m\,\underline{v})\right]$

$[Nm]$ $\quad = m \frac{d}{dt}\left[(\underline{x} - \underline{z}) \times \underline{v}\right]$

$\quad = m \frac{d}{dt}\left[\underline{x} \times \underline{v} - \underline{z} \times \underline{v}\right]$

$|\underline{a} \times \underline{b}| =$ area



$$= m\left[\dot{\underline{x}} \times \underline{v} + \underline{x} \times \dot{\underline{v}} - \underline{z} \times \dot{\underline{v}}\right]$$

$$= m\left[\cancelto{0}{\underline{v} \times \underline{v}} + \underline{x} \times \underline{a} - \underline{z} \times \underline{a}\right]$$

$$= m(\underline{x} - \underline{z}) \times \underline{a}$$

$$\tau = (\underline{x} - \underline{z}) \times m\underline{a} = \boxed{(\underline{x} - \underline{z}) \times \underline{f}}$$

Note: Torque = moment of force = moment

Types of Forces in Continuum Mechanics

I, Body Force

any force not due to physical contact

Example: gravitational body force

$$\underline{b}_g = \rho \underline{g} \qquad \frac{M}{L^3}\frac{L}{T^2} = \frac{M}{L^2T^2}$$

$\Rightarrow$ body force has units of $\frac{\text{force}}{\text{volume}}$

Net or resultant body force

$$\boxed{\underline{r}_b[B] = \int_B \underline{b}(\underline{x})\, dV}$$

units of force

Net or resultant torque on body around $\underline{z}$

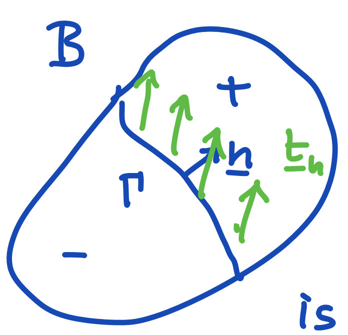
$$\boxed{\underline{T}_b[B] = \int_B (\underline{x}-\underline{z}) \times \underline{b} \; dV}$$

II Surface / Contact Forces

arise from physical contact

can be external and internal

Traction field



The force per unit area exerted by material on pos. side upon the material on neg. side is given by the traction field $\underline{t}_n$ for Γ.

The resultant force due to traction:

$$\boxed{\underline{r}_s[\Gamma] = \int_\Gamma \underline{t}_n(x) \, dA}$$

The resultant traction:

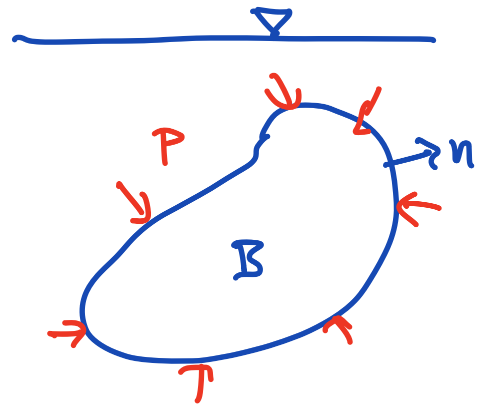
$$\boxed{\underline{T}_s[\Gamma] = \int_\Gamma (\underline{x} - \underline{z}) \times t_n(\underline{x}) \; dA}$$

Example:

Pressure, p,

$$\boxed{\underline{t} = -p\, \underline{n}}$$

hydrostatic surface force



Weight: Resultant grav. body force

$$\underline{f}_G = \underline{r}_b[B] = \int_B \rho_B\, g(\underline{x})\, dV \qquad \text{if } \rho \text{ \& } g \text{ are const}$$

$$= g \int_B \rho\, dV = m g$$

Acceleration of a free falling body (in vacuum)

$$f_G = m_B\, \underline{a} = m g \Rightarrow \underline{a}_g = g$$

Q: Where does $\underline{f}_G$ act on B?

Moment of Gravity

Moment = torque about origin

$$\boxed{\underline{\tau}_G = \underline{\tau}_b = \int \underline{x} \times \rho(\underline{x})\, g \, dV}$$

Resultant torque about $\underline{x}_m$ $\quad g = \text{const}$

$$\underline{\tau}_b = \int (\underline{x} - \underline{x}_m) \times \rho g \, dV \qquad \underline{x}_m = \text{const}$$

$$= \int_B \underline{x} \times \rho g - \underline{x}_m \times \rho g \, dV$$

$$= \underbrace{\int_B \underline{x} \rho \, dV}_{\underline{x}_m m_b} \times g - \int_B \underline{x}_m \rho \, dV \times g$$

$$- \underbrace{\int \rho \, dV}_{m_b} \, \underline{x}_m \times g$$

$$= \underline{x}_m \times m_b g - \underline{x}_m \times m_b g = \underline{0}$$

$\Rightarrow$ torque around $\underline{x}_m$ vanishes

Simplify "moment of gravity"

$$\tau_G = \int_B \underline{x} \times \rho g \, dV$$

$$= \int_B (\underline{x} - \underline{x}_m + \underline{x}_m) \times \rho g \, dV$$

$$= \int_B \cancelto{0}{(\underline{x} - \underline{x}_m)} \times \rho g \, dV + \int \underline{x}_m \times \rho g \, dV$$

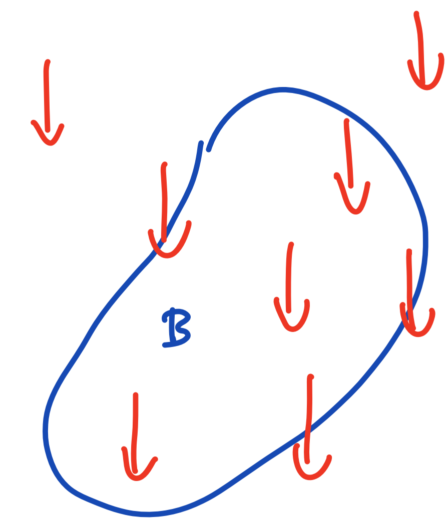
$$\tau_G = \int \underline{x}_m \times \rho g \, dV = \underline{x}_m \times g \underbrace{\int \rho \, dV}_{m}$$

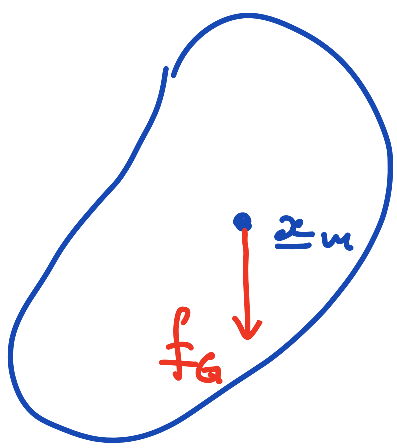
$$\boxed{\tau_G = \underline{x}_m \times (m_b g)}$$

Moment of Gravity
(Torque around origin)

Continuum

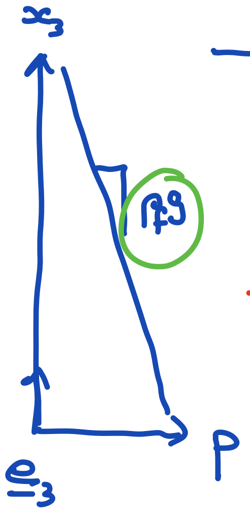
Discrete

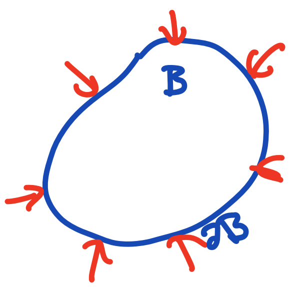




$g(\underline{x})$

Buoyancy: Resultant hydrostatic surface force





Archimedes principle:

submerged body is buoyed up by force equal to weight of displaced water.

Buoyancy ⇒ resultant hydrostatic surface force

$$\underline{F}_s = \underline{f}_B = \int_{\partial B} \underline{t}\, dA = -\int_{\partial B} p\, \underline{n}\, dA$$

need to convert to volume integral.

Gradient theorem $\boxed{\int_{\partial\Omega} \phi(\underline{x})\, \underline{n}\, dA = \int_{\Omega} \nabla\phi\, dV}$

$$\underline{f}_B = -\int_B \nabla p\, dV \qquad p = -\rho_f g x_3 \qquad g = |\underline{g}|$$

$$\nabla p = -\rho_f g\, \underline{e}_3$$

$$\underline{f}_B = -\int \rho_f \underline{g}\, dV \qquad \nabla p = \rho_f \underline{g} \qquad \nabla x_3 = \underline{e}_3$$

$$\underline{g} = -g\, \underline{e}_3$$

$$\boxed{\underline{f}_B = -g \int \rho_f \, dV = -g\, m_f}$$

Moment of Buoyancy

torque of hydrostatic surface force around origin

Can show τ_s vanishes around center of mass of fluid $\Rightarrow$ center of volume of floating body

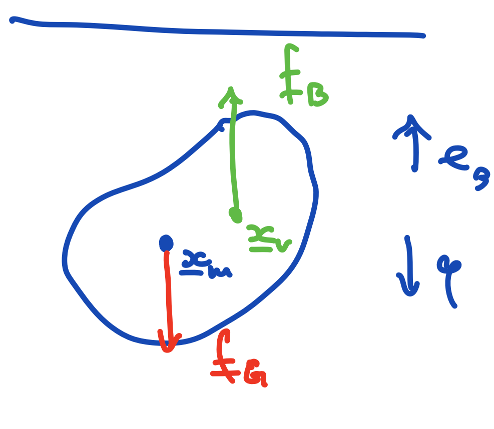
$$\underline{\tau}_s = \int_{\Gamma} (\underline{x} - \underline{x}_v) \times \underline{t}_n \, dS = \underline{0}$$

it is simple to show that

$$\boxed{\underline{\tau}_B = -\underline{x}_v \times (m_f\, \underline{g})}$$

Hydrostatic equilibrium

$$\underline{f} = \underline{f}_G + \underline{f}_B = -\int_B \rho_b\, g\, \underline{e}_3 \, dV - \int_{\partial B} p\, \underline{n} \, dS$$



$$\underline{f} = \int (\rho_f - \rho_b)\, g\, \underline{e}_3 \, dV = (m_f - m_b)\, g\, \underline{e}_3$$

$$= (m_b - m_f)\, \underline{g}$$

Hydrostatic eqbm

$$\boxed{\underline{f} = (m_b - m_f)\, \underline{g} = 0}$$