Lecture 20: Rates and Reynolds Transport Theorem

Logistics:
- HW5 grades & feedback uploaded
- HW6 please turn it in!
- HW7 is due
- HW8 will be posted

Last time:
- Motions $\varphi(\underline{X}, t)$
- material & spatial descriptions
- 3 different time derivatives:

1) Material derivative of material field

$$\frac{\partial}{\partial t} \Omega(\underline{X}, t) = \left.\frac{\partial \Omega}{\partial t}\right|_{\underline{X}} = \dot{\Omega}(\underline{X}, t)$$

2) Spatial time derivative of a spatial field

$$\frac{\partial}{\partial t} \Gamma(\underline{x}, t) = \left.\frac{\partial \Gamma}{\partial t}\right|_{\underline{x}}$$ (local derivative)

3) Material derivative of spatial field

$$\dot{\Gamma}(\underline{x}, t) = \frac{\partial}{\partial t} \Gamma(\varphi(\underline{X}, t), t) = \boxed{\frac{\partial \Gamma}{\partial t} + \underline{v} \cdot \nabla \Gamma}$$

Today:
- Rate of deformation tensors
- Reynolds transport theorem
- Derivatives of tensor functions

$$J = J(\underline{F}) \quad \longrightarrow \quad \dot{J}(F(\underline{X}, t))$$

Rate of deformation tensor

"strain rate tensor"

so far deformation/displacement gradients

$$\underline{\underline{F}} = \nabla \varphi \qquad \underline{\underline{H}} = \nabla \underline{u}$$

Here we are interested in velocity gradient:

Material velocity gradient $\nabla_X \underline{V}$

$\underline{\underline{F}} = \nabla_X \varphi \qquad \varphi_{i,j} \qquad \& \qquad \underline{V} = \dot{\varphi} \qquad V_i = \varphi_{i,t}$

$$\dot{\underline{\underline{F}}} = \frac{\partial}{\partial t}(\nabla_X \varphi) = \nabla_X \left(\frac{\partial}{\partial t}\varphi\right) = \nabla_X \dot{\varphi} = \nabla_X \underline{V}$$

$$\boxed{\nabla_X \underline{V} = \dot{\underline{\underline{F}}}}$$

Note analogy:

$$\varphi(\underline{X} + \Delta X, t) \approx \varphi(\underline{X}, t) + \overbrace{\nabla \varphi}^{\underline{\underline{F}}} \underline{\underline{\Delta X}}$$

take material time deriv.

$$\dot{\varphi}(\underline{X} + \Delta X, t) \approx \dot{\varphi}(\underline{X}, t) + \dot{\underline{\underline{F}}} \, \Delta X$$

$$\Rightarrow \quad \underline{V}(\underline{X} + \Delta X, t) \approx \underline{V}(\underline{X}, t) + \nabla_X \underline{V} \, \Delta X$$

$\Rightarrow \quad \nabla_X V$ takes role $\underline{\underline{F}}$ in mat. velocity expans.

Spatial velocity gradient $\nabla_x \underline{v} = \underline{\underline{l}}$

Note: $\underline{V}(\underline{X},t) = \underline{v}(\varphi(\underline{X},t),t)$

same vector field once expressed

in terms of $\underline{X}$ and $\underline{x}$

but $\nabla_X \underline{V} \neq \nabla_x \underline{v} \big|_{x=\varphi(X,t)}$

because derivatives are in different direction

What is relation between $\nabla_x \underline{v}$ and $\nabla_X \underline{V}$?

$\underline{V}(\underline{X},t) = \underline{v}(\varphi(\underline{X},t),t)$

$\dot{F}_{ij} = \frac{\partial}{\partial X_j} V_i = \underset{\uparrow}{\frac{\partial}{\partial X_j}} v_i(\varphi(\underline{X},t),t)$

$\frac{\partial}{\partial X_j} = \frac{\partial}{\partial x_k} \underbrace{\frac{\partial x_k}{\partial X_j}}_{F_{kj}} = \frac{\partial}{\partial x_k} F_{kj}$

substitute

$\dot{F}_{ij} = \frac{\partial}{\partial x_k} v_i(\underline{x},t)\, F_{kj} = \underbrace{v_{i,k}}_{\nabla_x v} F_{kj}$

$\Rightarrow \nabla_X \underline{\underline{V}} = \dot{\underline{\underline{F}}} = \nabla_x v\, \underline{\underline{F}} = \underline{\underline{l}}\, \underline{\underline{F}}$

solve for $\underline{\underline{l}}$

$$\boxed{\underline{\underline{l}} = \nabla_x \underline{v} = \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}}$$

$$\boxed{\nabla_X \underline{V} = \nabla_x \underline{v}\,\underline{\underline{F}}}$$

To understand $\underline{\underline{l}}$ need to decompose it

$$\boxed{\begin{aligned} \underline{\underline{l}} &= \underline{\underline{d}} + \underline{\underline{w}} \\ \underline{\underline{d}} &= \mathrm{sym}(\underline{\underline{l}}) = \tfrac{1}{2}\left(\nabla_x \underline{v} + \nabla_x \underline{v}^T\right) \\ \underline{\underline{w}} &= \mathrm{skew}(\underline{\underline{l}}) = \tfrac{1}{2}\left(\nabla_x \underline{v} - \nabla_x \underline{v}^T\right) \end{aligned}}$$

$\underline{\underline{d}}$ = rate of strain tensor

$\underline{\underline{w}}$ = spin tensor

Interpretation of $\underline{\underline{d}}$ and $\underline{\underline{w}}$:

$$\underline{v}(\underline{x} + \underline{\Delta x}, t) \approx \underline{v}(\underline{x}, t) + \nabla_x \underline{v}\,\Delta \underline{x}$$

$$\approx \underline{v}(\underline{x}, t) + (\underline{\underline{d}} + \underline{\underline{w}})\,\Delta \underline{x}$$

$$\approx \underline{v}(\underline{x}, t) + \underline{\underline{d}}\,\underline{\Delta x} + \underline{\underline{w}}\,\Delta \underline{x}$$

because $\underline{\underline{w}}$ is skew: $\underline{\underline{w}}\,\underline{\Delta x} = \underline{w} \times \underline{\Delta x}$

where $\underline{w}$ is axial vector

$$\underline{v}(\underline{x}+\Delta\underline{x},t) = \underline{v}(\underline{x},t) + \underline{\underline{d}}\,\Delta\underline{x} + \underline{w} \times \Delta\underline{x}$$

$\Rightarrow$ $\underline{\underline{d}}$ sym $\rightarrow$ is strech rate

$\Rightarrow$ $\underline{\underline{w}}$ skew $\rightarrow$ is rate of change in orientation (spin)

ω = angular velocity

$\mathrm{vec}(\nabla_x = |\nabla_x \times \underline{v}| = 2\omega$

what is relation $\underbrace{\nabla_x \times \underline{v}}_{\text{vector}}$ and $\underbrace{\mathrm{skew}(\nabla_x \underline{v})}_{\text{axial vector}}$

hypothesis:

$$\underbrace{|\mathrm{vec}(\mathrm{skew}(\nabla_x \underline{v}))|}_{\text{axial vec.}} = |\nabla_x \times \underline{v}| = 2\underbrace{\omega}_{\text{angular velocity}}$$

End of Kinematics

$\Rightarrow$ Balance laws

Reynolds Transport Theorem

$\varphi(\underline{X},t)$ and $\underline{v}(\underline{x},t)$ Ω_t arbitrary volume

$$\boxed{\frac{d}{dt}\int_{\Omega_t}\phi\, dV_x = \int_{\Omega_t}\frac{\partial\phi}{\partial t}\, dV_x + \oint_{\partial\Omega_t}\phi\,\underline{v}\cdot n\, dA_x}$$

Key: Ω_t depends on φ we can compute this without knowledge of φ

Difficulty is Ω_t changes with time

$\Rightarrow$ move to reference configuration Ω_0:

$$\frac{d}{dt}\int_{\Omega_t}\phi(\underline{x},t)\, \underset{\substack{\uparrow\\ J(\underline{X},t)\, dV_X}}{dV_x} = \frac{d}{dt}\int_{\Omega_0}\underbrace{\phi(\varphi(\underline{X},t),t)}_{\phi_m(\underline{X},t)}\, J(\underline{X},t)\, dV_X$$

Ω_0 is fixed $\rightarrow$ exchange $\int$ and $\frac{d}{dt}$

$$= \int_{\Omega_0}\frac{d}{dt}\left(\phi_m(\underline{X},t)\, J(\underline{X},t)\right) dV_X$$

$$= \int_{\Omega_0}\dot{\phi}_m J + \phi_m \dot{J}\, dV_X$$

where $\quad \dot{J} = J(\nabla_x \cdot \underline{v})_m \quad \rightarrow$ show later

$$= \int_{\Omega_0} \dot{\phi}_m J + \phi_m J (\nabla_x \cdot \underline{v})_m \, dV_X$$

$$= \int_{\Omega_0} \Big(\underbrace{\dot{\phi}_m + \phi_m (\nabla_x \cdot \underline{v})_m}_{f_m} \Big) \underbrace{J \, dV_X}_{dV_x}$$

$$= \int_{\Omega_t} \dot{\phi}_m + \phi_m \ \nabla_x \cdot \underline{v} \ \ dV_x$$

$$\uparrow$$

$$\dot{\phi} = \frac{\partial \phi}{\partial t} + \underline{v} \cdot \nabla_x \phi$$

$$= \int_{\Omega_t} \frac{\partial \phi}{\partial t} + \underbrace{\underline{v} \cdot \nabla_x \phi + \phi \ \nabla_x \cdot \underline{v}}_{\nabla_x \cdot (\phi \underline{v})} \ dV_x$$

$$= \int_{\Omega_t} \frac{\partial \phi}{\partial t} + \nabla_x \cdot (\phi \underline{v}) \ dV_x$$

$$= \int_{\Omega_t} \frac{\partial \phi}{\partial t} dV_x + \int_{\Omega_t} \nabla_x \cdot (\phi \underline{v}) \, dV_x$$

divergence theor

$$= \int_{\Omega_t} \frac{\partial \phi}{\partial t} dV_x + \oint_{\partial \Omega_t} \phi \, \underline{v} \cdot \underline{n} \, dA_x \qquad \underline{n} = \text{outward normal}$$

Where did $\dot{J} = J(\nabla \cdot \underline{v})_m$ come from?

Derivatives of tensor functions

So far we have considered field: $\phi(\underline{x})$, $\underline{v}(\underline{x})$, $\underline{\underline{S}}(\underline{x})$

$\Rightarrow$ vector functions because they are functions of a vector.

Here we have tensor functions:

- scalar-valued tensor functions: $\psi = \psi(\underline{\underline{S}})$
- tensor-valued tensor functions: $\underline{\underline{\Sigma}} = \underline{\underline{\Sigma}}(\underline{\underline{S}})$

Derivatives of scalar valued tensor functions:

Typical examples: $\det(\underline{\underline{A}})$ $\quad \mathrm{tr}(\underline{\underline{A}})$

Definition: $\psi(\underline{\underline{S}})$ is differentiable at $\underline{\underline{A}}$ if there exists a tensor $D\psi(\underline{\underline{A}})$ s.t

$$\psi(\underline{\underline{A}} + \underline{\underline{H}}) \approx \psi(\underline{\underline{A}}) + D\psi(A) : \underline{\underline{H}} + \text{h.o.t}$$

or $\underline{\underline{H}} = \epsilon \underline{\underline{U}}$

$$D\psi(\underline{\underline{A}}) : \underline{\underline{U}} = \frac{d}{d\epsilon} \psi(\underline{\underline{A}} + \epsilon \underline{\underline{U}})\Big|_{\epsilon=0}$$

$D\psi(\underline{\underline{A}})$ is derivative of ψ at $\underline{\underline{A}}$

$\{\underline{e}_i\}$ $\quad D\psi(\underline{\underline{A}}) = \frac{\partial \psi}{\partial A_{ij}} \underline{e}_i \otimes \underline{e}_j$

$\Rightarrow$ can be shown by component wise → see note

Derivative of trace:

$\psi(\underline{\underline{A}}) = tr(\underline{\underline{A}}) = A_{ii}$

$$D tr(\underline{\underline{A}}) = \frac{\partial A_{ii}}{\partial A_{kl}} \underline{e}_k \otimes \underline{e}_l = \delta_{ik} \delta_{il} \underline{e}_k \otimes \underline{e}_l = \underline{e}_i \otimes \underline{e}_i =$$
$$= \underline{\underline{I}}$$

$$\boxed{D tr(\underline{\underline{A}}) = \underline{\underline{I}}}$$