Lecture 21: Balance laws

Logistics: - HW6 need on last submission

- HW7 please turn in

Last time: Rates of deformation

- Velocity gradient: $\nabla_X \underline{V} = \dot{\underline{\underline{F}}}$

$$\nabla_x \underline{v} = \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1} = \underline{\underline{\ell}}$$

- Symmetric - Skew decomposition:

$$\underline{\underline{\ell}} = \underline{\underline{d}} + \underline{\underline{\ell}}$$

$$\underline{\underline{d}} = \frac{1}{2}\left(\nabla \underline{\underline{v}} + \nabla \underline{v}^T\right) \qquad \text{rate of strain tensor}$$

$$\underline{\underline{w}} = \frac{1}{2}\left(\nabla \underline{v} - \nabla \underline{v}^T\right) \qquad \text{spin tensor}$$

- Reynolds Transport Theorem

$$\frac{d}{dt}\int_{\Omega_t} \phi \, dV_x = \int_{\Omega_t} \frac{\partial \phi}{\partial t}\, dV_x + \oint_{\partial\Omega} \phi \, \underline{v} \cdot \underline{n} \, dA_x$$

- Derivatives of tensor functions $\qquad \dot{J}$

$$D\Psi(\underline{A}) = \frac{\partial \Psi}{\partial A_{ij}} \underline{e}_i \otimes \underline{e}_j \qquad J(\underline{\underline{F}})$$

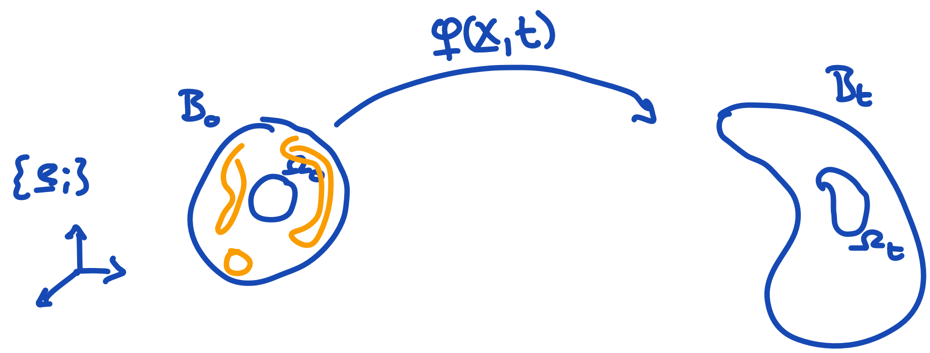
Today: - Balance laws

Local Eulerian Balance Laws

local = no integrals

Eulerian = $\underline{x}$ spatial description

(Lagrangian : $\underline{X}$ material description)



$$\Omega_t = \varphi(\Omega_0)$$

Ω_0 is arbitrary

I, Conservation of mass

Mass in Ω_t : $M[\Omega_t] = \int_{\Omega_t} \rho(\underline{x},t)\, dV_{\underline{x}}$

in absence of reactions or relativistic effects

$\Rightarrow$ mass is conserved

$$\boxed{\frac{d}{dt} M[\Omega_t] = 0} \quad \Rightarrow \quad M[\Omega_t] = M[\Omega_0]$$

using $dV_x = J(\underline{X}, t)\, dV_X$ $\qquad J = \det(\underline{\underline{F}}) > 0$

$$M[\Omega_0] = M[\Omega_t] = \int_{\Omega_t} \rho(\underline{x}, t)\, dV_x = \int_{\Omega_0} \underbrace{\rho(\underline{\varphi}(\underline{X}, t), t)}_{\rho_m(\underline{X}, t)} J(\underline{X}, t)\, dV_X$$

$$M[\Omega_0] = \int_{\Omega_0} \rho_m(\underline{X}, t)\, J(\underline{X}, t)\, dV_X$$

at $t = 0$: $\Omega_t \to \Omega_0$ $\quad J(\underline{X}, 0) = 1$ $\quad \underline{x} = \underline{X}$

$$M[\Omega_0] = \int_{\Omega_0} \rho_m(\underline{X}, 0)\, dV_X = \int_{\Omega_0} \rho_0(\underline{X})\, dV_X$$ $\qquad \rho_0$ = initial mass

$$\Rightarrow \int_{\Omega_0} \rho_m(\underline{X}, t)\, J(\underline{X}, t) - \rho_0(\underline{X})\, dV_X = 0$$

by arbitraryness of Ω_0 $\rightarrow$ integrand must be zero

$$\boxed{\rho_m(\underline{X}, t)\, J(\underline{X}, t) = \rho_0(\underline{X})}$$

Lagrangian statement of mass conservation

$$J = \frac{dV_x}{dV_X}$$

$$\frac{dV_x}{dV_X} = \frac{\rho_0}{\rho_m}$$

Convert to Eulerian : $\frac{\partial}{\partial t}$

$$\frac{\partial}{\partial t}\left(\rho_m(X,t)\, J(\underline{X},t)\right) = \frac{\partial}{\partial t}\rho_0(X) = 0$$

$$\left(\frac{\partial}{\partial t}\rho_m(\underline{X},t)\right) J(\underline{X},t) + \rho_m \underbrace{\frac{\partial}{\partial t} J(\underline{X},t)}$$ $= 0$

$$\dot{\rho}_m(\underline{X},t)\, \cancel{J}(X,t) + \rho_m\, \cancel{J}(\nabla_x \cdot \underline{v})_m = 0$$

switch to spatial descripion $\rho_m(\underline{X},t) = \rho(x,t)$

$x = \varphi(X,t)$

$$\dot{\rho}(\underline{x},t) + \rho(\underline{x},t)\,\nabla_x \cdot \underline{v}(\underline{x},t) = 0$$

$\Rightarrow$ $\boxed{\dot{\rho} + \rho\,\nabla\cdot\underline{v} = 0}$ local Eulerian descripion

$$\dot{\rho} = \frac{\partial \rho}{\partial t} + \underline{v}\cdot\nabla\rho$$

$$\frac{\partial \rho}{\partial t} + \underbrace{\underline{v}\cdot\nabla\rho + \rho\,\nabla\cdot\underline{v}}_{\nabla\cdot(\rho\underline{v})} = 0$$

$\Rightarrow$ $\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\underline{v}) = 0}$ conservative local Eulerian form

Time derivative of integrals with respect to mass

$$\boxed{\frac{d}{dt}\int_{\Omega_t} \phi(\underline{x},t)\,\rho(\underline{x},t)\,dV_x = \int_{\Omega_t} \dot{\phi}(\underline{x},t)\,\rho(\underline{x},t)\,dV_x}$$

$$\int_{\Omega_t} \phi(\underline{x},t)\,\rho(\underline{x},t)\,dV_x = \int_{\Omega_0} \phi_m(\underline{X},t)\underbrace{\rho_m(\underline{X},t)\,J(\underline{X},t)}_{\rho_0(X)}\,dV_X$$

$$\int_{\Omega_t} \qquad = \int_{\Omega_0} \phi_m(\underline{X},t)\,\rho_0(\underline{X})\,dV_X$$

$$\frac{d}{dt}\int_{\Omega_0} \phi_m(\underline{X},t)\,\rho_0(\underline{X})\,dV_X = \int_{\Omega_0} \dot{\phi}_m(\underline{X},t)\,\rho_0(\underline{X})\,dV_X$$

$$= \int_{\Omega_0} \dot{\phi}_m(\underline{X},t)\,\underline{\rho_m(X,t)\,J(X,t)}\,dV_X$$

$$= \int_{\Omega_t} \dot{\phi}(\underline{x},t)\,\rho(\underline{x},t)\,dV_x$$

<u>Laws of inertia</u> (Galileo & Newton)

linear momentum: $\boxed{\underline{l}[\Omega_t] = \int_{\Omega_t} \rho(\underline{x},t)\,\underline{v}(\underline{x},t)\,dV_x}$

angular momentum: $\underline{j}[\Omega_t]_{\underline{z}} = \int_{\Omega_t} (\underline{x}-\underline{z}) \times \rho(\underline{x},t)\,\underline{v}(\underline{x},t)\,dV_x$

In fixed frame of reference the change in momentum is equal to resultant force/torque.

$$\boxed{\begin{aligned} \frac{d}{dt}\underline{L}[\Omega_t] &= \int_{\Omega_t} \rho(\underline{x},t)\,\underline{b}(\underline{x},t)\,dV_x + \int_{\partial\Omega_t} \underline{t}(\underline{x},t)\,dA_x \\ \frac{d}{dt}\underline{j}[\Omega_t]_{\underset{\uparrow\,\underline{x}=\underline{0}}{0}} &= \int_{\Omega_t} \underline{x} \times \rho(\underline{x},t)\,\underline{b}(\underline{x},t)\,dV_x + \int_{\partial\Omega_t} \underline{x} \times \underline{t}(\underline{x},t)\,dA_x \end{aligned}}$$

Balance of linear momentum

Cauchy stress: $\underline{t} = \underline{\underline{\sigma}}\,\underline{n}$

$$\frac{d}{dt}\int_{\Omega_t} \rho\,\underline{v}\,dV_x = \int_{\partial\Omega_t} \underline{\underline{\sigma}}\,\underline{n}\,dA_x + \int_{\Omega_t} \rho\,\underline{b}\,dV_x$$

use tensor divergence theorem

$$\frac{d}{dt}\int_{\Omega_t} \rho\,\underline{v}\,dV_x = \int_{\Omega_t} \nabla\cdot\underline{\underline{\sigma}}\,dV_x + \int_{\Omega_t} \rho\,\underline{b}\,dV_x$$

using derivative with respect to mass

$$\int_{\Omega_t} \rho\,\dot{\underline{v}} - \nabla\cdot\underline{\underline{\sigma}} - \rho\,\underline{b}\;dV_x = 0$$

by arbitraryness Ω_t $\rightarrow$ localize

$$\Rightarrow \quad \boxed{\rho\,\dot{\underline{v}} - \nabla\cdot\underline{\underline{\sigma}} = \rho\,\underline{b}}$$

local Eulerian form of lin. mom. balance.

Cauchy's first equation of motion

Rewrite this in conservative form.

$$\rho \dot{\underline{v}} = \rho\left(\frac{\partial \underline{v}}{\partial t} + (\nabla_x \underline{v})\underline{v}\right) = \rho \frac{\partial \underline{v}}{\partial t} + \rho (\nabla_x \underline{v})\underline{v}$$

$$= \frac{\partial}{\partial t}(\rho \underline{v}) - \frac{\partial \rho}{\partial t}\underline{v} + \rho(\nabla_x \underline{v})\underline{v}$$

mass balance $\quad \frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho \underline{v})$

$$\rho \dot{\underline{v}} = \frac{\partial}{\partial t}(\rho \underline{v}) + \nabla\cdot(\rho \underline{v})\,\underline{v} + (\nabla_x \underline{v})\,(\rho \underline{v})$$

use $\quad \nabla\cdot(\underline{a}\otimes\underline{b}) = (\nabla \underline{a})\,\underline{b} + \underline{a}\,\nabla\cdot\underline{b} \qquad$ (HW 6)

$$\underline{b} = \rho \underline{v} \qquad \underline{a} = \underline{v}$$

$$\rho \dot{\underline{v}} = \frac{\partial}{\partial t}(\rho \underline{v}) + \nabla\cdot(\rho \underline{v}\otimes\underline{v})$$

substitute into lin. mom. bal.

$$\boxed{\frac{\partial}{\partial t}(\rho \underline{v}) + \nabla_x\cdot(\rho \underline{v}\otimes\underline{v} - \underline{\underline{\sigma}}) = \rho \underline{b}}$$

conservative local Eulerian form

conserved quantity: $\underline{L} = \rho \underline{v}$

advective momentum flux: $\rho \underline{v}\otimes\underline{v}$

diffusive momentum flux: $-\underline{\underline{\sigma}}$

$\underline{v} = \underline{0} \quad \Rightarrow \quad -\nabla\cdot\underline{\underline{\sigma}} = \rho b \qquad$ mechanical eqbm

Balance of angular momentum

$$\frac{d}{dt}\int_{\Omega_t} \underline{x} \times \rho \underline{v} \, dV_x = \int_{\partial\Omega_t} \underline{x} \times \underline{t} \, dA_x + \int_{\Omega_t} \underline{x} \times \rho \underline{b} \, dV_x$$

lhs:

$$\frac{d}{dt}\int_{\Omega_t} \underline{x} \times \rho \underline{v} \, dV_x = \frac{d}{dt}\int_{\Omega_t} \rho (\underline{x} \times \underline{v}) \, dV_x$$

$$= \int_{\Omega_t} \rho (\underbrace{\cancel{\dot{\underline{x}} \times \underline{v}}}_{\underline{v} \times \underline{v}}{}^{0} + \underline{x} \times \dot{\underline{v}}) \, dV_x \qquad \dot{\underline{x}} = \underline{v}$$

$$= \int_{\Omega_t} \rho (\underline{x} \times \dot{\underline{v}}) \, dV_x$$

substitute Cauchy stress $\underline{t} = \underline{\underline{\sigma}}\,\underline{n}$ into r.h.s.

$$\int_{\Omega_t} \rho (\underline{x} \times \dot{\underline{v}}) \, dV_x = \int_{\partial\Omega} \underline{x} \times \underline{\underline{\sigma}}\,\underline{n} \, dA_x + \int_{\Omega_t} \rho (\underline{x} \times \underline{b}) \, dV_x$$

$$\int_{\Omega_t} \underline{x} \times (\underbrace{\rho \dot{\underline{v}} - \rho \underline{b}}_{\nabla_x \cdot \underline{\underline{\sigma}}}) \, dV_x = \int_{\partial\Omega} \underline{x} \times \underline{\underline{\sigma}}\,\underline{n} \, dA_x$$

from lin. mom. balance

$$\Rightarrow \quad \boxed{\int_{\Omega_t} \underline{x} \times \nabla_x \cdot \underline{\underline{\sigma}} \, dV_x = \int_{\partial\Omega} \underline{x} \times \underline{\underline{\sigma}}\,\underline{n} \, dA_x}$$

This is exactly statement for the static mechanical eqblm $\Rightarrow$ Lecture 14

$\Rightarrow$ $\boxed{\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T}$ extends to transient cases!