Lecture 22: Energy balance

Logistics: HW6 still missing one

→ next Tuesday 9:30 am last chance

HW7 still missing two

Last time: local Eulerian balance laws

mass: $\dot{\rho} + \rho \nabla \cdot \underline{v} = 0$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\underbrace{\rho \underline{v}}_{\text{advective mass flux}}) = 0 \qquad \text{conservative}$$

lin. mom.: $\rho \dot{\underline{v}} - \nabla \cdot \underline{\underline{\sigma}} = \rho \underline{b}$

$$\frac{\partial}{\partial t}(\rho \underline{v}) + \nabla \cdot (\rho \underline{v} \otimes \underline{v} - \underline{\underline{\sigma}}) = \rho \underline{b}$$

ang. mom: $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$

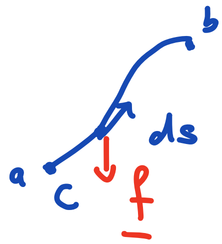
Today:
- Review of thermodynamics
- Work, heat, power, energy
- Local Eulerian balance
- stress power
- entropy

Work & Power

Work: energy transferred by application of force along a distance.

$W = F s$

$W = \int_C \underline{f} \cdot d\underline{s} = \int_{t_1}^{t_2} \underline{f} \underbrace{\frac{d\underline{s}}{dt}}_{\underline{v}} dt$



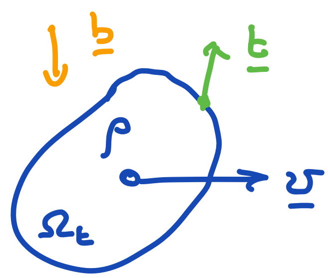
$W = \int_{t_1}^{t_2} \underline{f} \cdot \underline{v}\, dt = \int \frac{dW}{dt} dt \Rightarrow \frac{dW}{dt} = \underline{f} \cdot \underline{v}$

Power is the rate of work

$$\boxed{P = \frac{dW}{dt} = \underline{f} \cdot \underline{v}}$$

Working rate of work being done ~ power

Exchange of mechanical energy between body and surrounding



Kinetic energy

$$\boxed{\mathcal{K}[\Omega_t] = \int_{\Omega_t} \frac{1}{2} \rho(\underline{x}, t) \, |\underline{v}(\underline{x}, t)|^2 \, dV_x}$$

Power of external forces on Ω_t

$$\mathcal{P}[\Omega_t] = \underline{r}[\Omega_t] \cdot \underline{v} = \underline{r}_b[\Omega_t] \cdot \underline{v} + \underline{r}_s[\Omega_t] \cdot \underline{v}$$

$$= \int_{\Omega_t} \rho\, \underline{b} \cdot \underline{v}\, dV_x + \int_{\partial\Omega_t} \underline{t} \cdot \underline{v}\, dA_x$$

Net working $\mathcal{W}[\Omega_t]$ of external forces on Ω_t

$$\boxed{\mathcal{W}[\Omega_t] = \mathcal{P}[\Omega_t] - \frac{d}{dt}\mathcal{K}[\Omega_t]}$$

$\mathcal{W}[\Omega_t] > 0$: mechanical energy is stored in body

$\mathcal{W}[\Omega_t] < 0$: " " " released from body

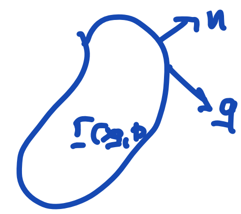
Temperature and heat

macroscopic quantity $\rightarrow$ magnitude of microscopic velocity fluctuations

absolute temp. field $\theta(\underline{x}, t) > 0$

Heat is energy associated with velocity fluctuations

$\Rightarrow$ Thermal energy



Heating is rate of heat gain / loss

I) body heating: $Q_b[\Omega_t] = \int_{\Omega_t} \rho \, r \, dV_x$

$r(x,t)$ = heat supply / loss per unit mass

II) surface heating: $Q_s[\Omega_t] = -\int_{\partial\Omega_t} q \cdot \underline{n} \, dA_x$

$q(x,t)$ = heat flux vector

Net heating

$$\boxed{Q[\Omega_t] = Q_b[\Omega_t] + Q_s[\Omega_t] = \int_{\Omega_t} \rho r \, dV - \int_{\partial\Omega} q \cdot n \, dA}$$

Internal energy and 1st law of thermodynamics

Internal energy is energy not associated with kinetic energy.

Internal energy = thermal (heat) + mechanical (elastic)

neglect electromagnetic & chemical energy

$$\boxed{\mathcal{U}[\Omega_t] = \int_{\Omega} \rho u \, dV_x}$$

$u(\underline{x},t)$ is internal energy density per unit mass

First law of Thermo:

$$\boxed{\frac{d}{dt}\mathcal{U}[\Omega_t] = Q[\Omega_t] + \mathcal{W}[\Omega_t]}$$

$$\mathcal{W}[\Omega_t] = \mathcal{P}[\Omega_t] - \frac{d}{dt}\mathcal{K}[\Omega_t]$$

$$\Rightarrow \frac{d}{dt}\Big(\underbrace{\mathcal{U}[\Omega_t] + \mathcal{K}[\Omega_t]}_{\text{total energy}}\Big) = Q[\Omega_t] + \mathcal{P}[\Omega_t]$$

In some cases power of an external force can be written $\mathcal{P}[\Omega_t] = -\frac{d}{dt} G[\Omega_t]$

$$\frac{d}{dt}\left(\mathcal{U}[\Omega_t] + \mathcal{K}[\Omega_t] + G[\Omega_t]\right) = Q[\Omega_t]$$

Balance of Energy in local Eulerian form

Net working in Eulerian form?

Power: $\mathcal{P} = \underline{f} \cdot \underline{v}$

Newton's 2nd law: $\underline{f} = m\,\underline{a} = m\,\underline{\dot{v}}$

lin. mom. balance: $\rho\,\underline{\dot{v}} = \nabla \cdot \underline{\underline{\sigma}} + \rho\,\underline{b}$

dot product $(\rho\,\underline{\dot{v}}) \cdot \underline{v}$:

$$\rho\,\underline{v} \cdot \underline{\dot{v}} = (\nabla \cdot \underline{\underline{\sigma}}) \cdot \underline{v} + \rho\,\underline{b} \cdot \underline{v}$$

integrate over Ω_t to identify $\mathcal{K}$, $\mathcal{P}$ ....

$$\int_{\Omega_t} \rho\,\underline{v} \cdot \underline{\dot{v}}\, dV_x = \int_{\Omega} \underline{(\nabla \cdot \underline{\underline{\sigma}}) \cdot \underline{v}} + \rho\,\underline{b} \cdot \underline{v}\, dV$$

use identity: $\nabla \cdot (\underline{\underline{A}}^T \underline{b}) = \underline{(\nabla \cdot \underline{\underline{A}}) \cdot \underline{b}} + \underline{\underline{A}} : \nabla \underline{b}$

$$(\nabla \cdot \underline{\underline{\sigma}}) \cdot \underline{v} = \nabla \cdot (\underline{\underline{\sigma}}^{\overset{\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T}{\cancel{T}}}\, \underline{v}) - \underline{\underline{\sigma}} : \nabla \underline{v}$$

substitute

$$\int_{\Omega_t} \rho\,\underline{v} \cdot \underline{\dot{v}}\, dV_x = -\int \underline{\underline{\sigma}} : \nabla \underline{v} + \underline{\nabla \cdot (\underline{\underline{\sigma}}\,\underline{v})} + \rho\,\underline{b} \cdot \underline{v}\, dV_x$$

Use property $\underline{\underline{S}} : \underline{\underline{D}} = \underline{\underline{S}} : \mathrm{sym}(\underline{\underline{D}})$ if $\underline{\underline{S}} = \underline{\underline{S}}^T$

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T \rightarrow \underline{\underline{\sigma}} : \nabla \underline{v} = \underline{\underline{\sigma}} : \mathrm{sym}(\nabla \underline{v}) = \underline{\underline{\sigma}} : \underline{\underline{d}}$$

$\underline{\underline{d}} \doteq \frac{1}{2}(\nabla \underline{v} + \nabla \underline{v}^T)$ rate of def. tensor

$$\int_{\Omega_t} \rho \underline{v} \cdot \dot{\underline{v}} \, dV_x = -\int_{\Omega_t} \underline{\underline{\sigma}} : \underline{\underline{d}} + \rho \underline{b} \cdot \underline{v} \, dV_x + \int_{\partial\Omega_t} \underline{\underline{\sigma}} \underline{v} \cdot \underline{n} \, dA_x$$

definition of transpose: $\underline{\underline{\sigma}} \underline{v} \cdot \underline{n} = \underline{v} \cdot \underline{\underline{\sigma}}^T \underline{n}$

$$= \underline{v} \cdot \underline{\underline{\sigma}} \underline{n} = \underline{v} \cdot \underline{t}$$

$$\int_{\Omega_t} \rho \underline{v} \cdot \dot{\underline{v}} \, dV_x = -\int_{\Omega_t} \underline{\underline{\sigma}} : \underline{\underline{d}} \, dV_x + \underbrace{\int_{\Omega} \rho \underline{b} \cdot \underline{v} \, dV + \int_{\partial\Omega_t} \underline{t} \cdot \underline{v} \, dA_x}_{\mathcal{P}[\Omega_t]}$$

rate of change of

identify l.h.s. as kinetic energy

$$\frac{d}{dt} \mathcal{K}[\Omega_t] = \frac{d}{dt} \int_{\Omega_t} \frac{1}{2} \rho \, \underline{v} \cdot \underline{v} \, dV_x = \frac{1}{2} \int_{\Omega_t} \rho \frac{d}{dt} (\underline{v} \cdot \underline{v}) \, dV_x$$

$$\frac{d}{dt} (v_i v_i) = \dot{v}_i v_i + v_i \dot{v}_i = 2 (v_i \dot{v}_i) = 2 \underline{v} \cdot \dot{\underline{v}}$$

substituting $\mathcal{K}$ & $\mathcal{P}$

$$\boxed{\frac{d}{dt} \mathcal{K}[\Omega_t] + \underbrace{\int_{\Omega_t} \underline{\underline{\sigma}} : \underline{\underline{d}} \, dV_x}_{\mathcal{W}[\Omega_t]} = \mathcal{P}[\Omega_t]}$$

$$\mathcal{W}[\Omega_t] = \mathcal{P}[\Omega_t] - \frac{d}{dt}\mathcal{K}[\Omega_t]$$ $$\boxed{\mathcal{W}[\Omega] = \int \underline{\underline{\sigma}} : \underline{\underline{d}}\, dV}$$

quantity $\underline{\underline{\sigma}} : \underline{\underline{d}}$ is called stress power associated with a motion and represents the rate of work by internal forces in a body.

Local Eulerian energy balance

Integral form of first law of thermo

$$\boxed{\frac{d}{dt}\mathcal{U}[\Omega_t] = \mathcal{Q}[\Omega_t] + \mathcal{W}[\Omega_t]}$$

where $\mathcal{U}[\Omega_t] = \int_{\Omega_t} \rho u \, dV_x$

$$\int_{\Omega_t} \nabla \cdot q \, dV$$

$$\mathcal{Q} = \int_{\Omega_t} \rho r \, dV_x - \underline{\int_{\partial\Omega_t} q \cdot n \, dA_x}$$

$$\mathcal{W} = \underline{\int_{\Omega_t} \underline{\underline{\sigma}} : \underline{\underline{d}} \, dV_x}$$

substituting

$$\frac{d}{dt}\int_{\Omega_t} \rho u \, dV = \int_{\Omega_t} \underline{\underline{\sigma}} : \underline{\underline{d}} + \rho r \, dV - \int_{\partial\Omega_t} \underline{q}\cdot n \, dA$$

deriv. with resp. towards

div. thm

$$\int_{\Omega_t} \left(\rho \dot{u} - \underline{\underline{\sigma}} : \underline{\underline{d}} + \nabla\cdot\underline{q} - \rho r\right) dV = 0$$

localize: $\boxed{\rho \dot{u} = \underline{\underline{\sigma}} : \underline{\underline{d}} - \nabla\cdot\underline{q} + \rho r}$

local Eulerian form of Energy balance

write in conservative form

$$\rho \dot{u} = \rho\left(\frac{\partial u}{\partial t} + \nabla u \cdot \underline{v}\right)$$

$$= \frac{\partial}{\partial t}(\rho u) - u \frac{\partial \rho}{\partial t} + \rho \nabla u \cdot \underline{v} \qquad \frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho \underline{v})$$

$$= \frac{\partial}{\partial t}(\rho u) + \underbrace{\nabla\cdot(\rho \underline{v})\, u + \rho \nabla u \cdot \underline{v}}$$

$$= \frac{\partial}{\partial t}(\rho u) + \nabla\cdot(\underline{v}\, \rho u)$$

$$\boxed{\frac{\partial}{\partial t}(\rho u) + \nabla\cdot(\underline{v}\,\rho u + \underline{q}) = \underline{\underline{\sigma}} : \underline{\underline{d}} + \rho r}$$

conservative form of energy bal.

In simple case:

$u = c_p T$ $\qquad$ c_p = heat capacity (at const pressure)

$q = -\kappa \nabla T$ $\qquad$ Fourier's law

κ: thermal conductivity

substitute

$$\frac{\partial}{\partial t}(\rho c_p T) + \nabla \cdot (\underline{v} \rho c_p T - \kappa \nabla T) = \underline{\underline{\sigma}} : \underline{\underline{d}} + \rho r$$

assume ρ, c_p, κ = const. simplify

$d = \frac{1}{2}[\nabla v + \nabla v^T]$

$$\rho c_p \frac{\partial T}{\partial t} + \nabla \cdot (\underbrace{\underline{v} \rho c_p T}_{\text{adv}} - \underbrace{\kappa \nabla T}_{\text{cond}}) = \underline{\underline{\sigma}} : \underline{\underline{d}} + \rho r$$

advection - conduction eqn

$\underline{v} = \underline{0}$

$\Rightarrow$ Heat equation

$$\rho c_p \frac{\partial T}{\partial t} = \kappa \underbrace{\nabla \cdot \nabla}_{\nabla^2} T + \rho r$$

$$\frac{\partial T}{\partial t} = \underbrace{\frac{\kappa}{\rho c_p}} \nabla^2 T + \frac{r}{c_p}$$

α = thermal diffusivity

$$\boxed{\frac{\partial T}{\partial t} = \alpha \nabla^2 T + \frac{\Gamma}{c_p}}$$ Heat equation