Lecture 23: Constitutive Theory

notes on rates don't work

Logistics: - HW 6 complete yey!

- HW 7 still outstanding
- HW 8 due Th

Last time: Energy balance

First law of Thermodynamics

$dU = dQ + dW$

$$\frac{d}{dt} \mathcal{U}[\Omega_t] = \underset{\uparrow\ \text{net heating}}{Q[\Omega_t]} + \underset{\uparrow\ \text{net working}}{W[\Omega_t]}$$

Heating: $q = -\kappa \nabla T$

$$Q[\Omega_t] = \int_{\Omega_t} \rho r \, dV - \int_{\partial\Omega_t} \underline{q} \cdot \underline{n} \, dV$$

Working

$$W[\Omega_t] = \mathcal{P}[\Omega_t] - \frac{d}{dt} K[\Omega_t] = \int_{\Omega} \underline{\underline{\sigma}} : \underline{\underline{d}} \, dV$$

Stress power: $\underline{\underline{\sigma}} : \underline{\underline{d}}$ $\qquad \underline{\underline{d}} = \frac{1}{2}(\nabla \underline{v} + \nabla \underline{v}^T)$

Today: Constitutive laws

relation between stress and strain

Constitutive Theory

Common constitutive laws:

Newtonian fluid: (incompressible) $\underline{\underline{\sigma}} = p\underline{\underline{I}} + \eta\,(\nabla\underline{v} + \nabla\underline{v}^T)$

$p = -\frac{1}{3}\,\mathrm{tr}(\underline{\underline{\sigma}})$ η = viscosity $\underline{v}$ = velocity

Linear elastic solid: (isotropic) $\underline{\underline{\sigma}} = \lambda(\nabla\cdot\underline{u})\underline{\underline{I}} + \mu\,(\nabla\underline{u} + \nabla\underline{u}^T)$

λ, μ = Lamé parameters $\underline{u}$ = displ.

Both derive from the same functional form

$$\boxed{\underline{\underline{G}}(\underline{\underline{A}}) = \underbrace{\mathbb{C}}_{4^{\text{th}}\text{ order tensor}}\,\underline{\underline{A}} = \lambda\,\mathrm{tr}(\underline{\underline{A}}) + 2\mu\,\mathrm{sym}(\underline{\underline{A}})}$$

linear elastic solid: $\underline{\underline{A}} = \nabla\underline{u}$

Newtonian fluid: $\underline{\underline{A}} = \nabla\underline{v}$

$\mathrm{tr}(\nabla\underline{u}) = \nabla\cdot\underline{u}$

$\Rightarrow$ for linear elastic solid just substitute

for Newtonian fluid there is a complication due to the incompressibility constraint.

Why do constitutive laws have this form?

Change in observer

Constitutive laws cannot depend on the observer

→ lecture on Change of basis

$\{\underline{e}_j\}$ and $\{\underline{e}'_i\}$ ⇒ change in basis tensor $\underline{\underline{Q}}$

$$\boxed{\begin{aligned} \underline{v} &= \underline{\underline{Q}}\,\underline{v}' \\ \underline{\underline{S}} &= \underline{\underline{Q}}\,\underline{\underline{S}}'\,\underline{\underline{Q}}^T \end{aligned}}$$

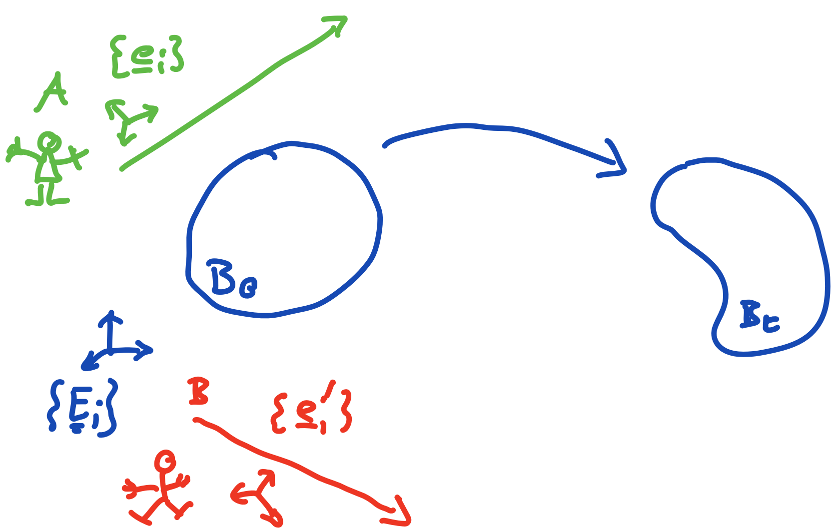
$\underline{\underline{Q}}$ = is a rotation

$\underline{\underline{Q}}\,\underline{\underline{Q}}^T = \underline{\underline{Q}}^T\underline{\underline{Q}} = \underline{\underline{I}}$

$\det(\underline{\underline{Q}}) = 1$

⇒ passive change in frame.

Change in observer → active change of frame



$\underline{x} = \underline{\varphi}(\underline{X}, t)$

$\underline{x}' = \underline{\varphi}'(\underline{X}, t)$

Note: Material frame is common to both observers

Note: both observers use same clock

Change in observer cannot induce deformation

$\Rightarrow$ two reference frames must be related by rigid body motion

$$\underline{x}' = \underline{\underline{Q}}(t)\,\underline{x} + \underline{c}(t)$$

$$\boxed{\underline{x}' = \underline{\underline{Q}}(t)\,\varphi(\underline{X},t) + \underline{c}(t)}$$ Eulerian transformation

Q = rotation $\qquad c$ = translation

$\Rightarrow$ constitutive laws cannot depend on observer

Effect on kinematic quantities:

$$\nabla\varphi = \underline{\underline{F}} \qquad \nabla\varphi' = \underline{\underline{F}}' = \underline{\underline{Q}}\,\underline{\underline{F}}$$

$$\underline{x}' = \varphi'(\underline{X},t) + c = Q(t)\,\varphi(\underline{X},t) + c(t)$$

Right Cauchy-Green Strain:

$$\underline{\underline{C}}' = \underline{\underline{F}}'^T\underline{\underline{F}}' = (\underline{\underline{F}}^T\underline{\underline{Q}}^T)(\underline{\underline{Q}}\,\underline{\underline{F}}) = \underline{\underline{F}}^T\underline{\underline{Q}}^T\underline{\underline{Q}}\,\underline{\underline{F}} = \underline{\underline{F}}^T\underline{\underline{F}} = \underline{\underline{C}}$$

$\Rightarrow$ not affected by rigid body motions.

material tensors $\rightarrow$ naturally objective

What about spatial tensors?

Axiom of frame indifference

Fields are called frame indifferent or objective if for any superimposed rigid body motion $\underline{x}' = \underline{\underline{Q}}(t)\,\underline{x} + \underline{c}$ we have

scalar field : $\phi'(\underline{x}',t) = \phi(\underline{x},t)$

vector field : $\underline{w}'(\underline{x}',t) = \underline{\underline{Q}}\,\underline{w}(\underline{x},t)$

Tensor field : $\underline{\underline{S}}'(\underline{x}',t) = \underline{\underline{Q}}\,\underline{\underline{S}}(\underline{x},t)\,\underline{\underline{Q}}^T$

Is $\nabla\underline{v}$ frame indifferent ?

$\underline{\underline{\ell}} = \nabla_x \underline{v} = \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}$ $\qquad$ $\underline{\underline{\ell}}' = \nabla_{x'}\underline{v}' = \dot{\underline{\underline{F}}}'\,\underline{\underline{F}}'^{-1}$

$\underline{\underline{F}}' = \underline{\underline{Q}}\,\underline{\underline{F}}$

Q: $\nabla_{x'}\underline{v}' = \underline{\underline{Q}}\,\nabla_x\underline{v}\,\underline{\underline{Q}}^T$?

$$\dot{\underline{\underline{F}}}' = \frac{d}{dt}(\underline{\underline{Q}}\,\underline{\underline{F}}) = \underline{\underline{Q}}\,\dot{\underline{\underline{F}}} + \dot{\underline{\underline{Q}}}\,\underline{\underline{F}}$$

$$\underline{\underline{F}}'^{-1} = (\underline{\underline{Q}}\,\underline{\underline{F}})^{-1} = \underline{\underline{F}}^{-1}\underline{\underline{Q}}^{-1} = \underline{\underline{F}}^{-1}\underline{\underline{Q}}^T$$

$$\underline{\underline{\ell}}' = \dot{\underline{\underline{F}}}'\underline{\underline{F}}'^{-1} = (\underline{\underline{Q}}\,\dot{\underline{\underline{F}}} + \dot{\underline{\underline{Q}}}\,\underline{\underline{F}})\,\underline{\underline{F}}^{-1}\underline{\underline{Q}}^T$$

$$= \underline{\underline{Q}}\,\underbrace{\dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}}_{\underline{\underline{\ell}}}\,\underline{\underline{Q}}^T + \dot{\underline{\underline{Q}}}\,\underbrace{\underline{\underline{F}}\,\underline{\underline{F}}^{-1}}_{\underline{\underline{I}}}\,\underline{\underline{Q}}^T$$

$\underline{\underline{\ell}}' = \underline{\underline{Q}}\,\underline{\underline{\ell}}\,\underline{\underline{Q}}^T + \dot{\underline{\underline{Q}}}\,\underline{\underline{Q}}^T$ $\Rightarrow$ no $\underline{\underline{\ell}} = \nabla_x v$ is objective
not

that's why $\nabla_x v$ is not use in constitutive laws.

The non-objective term $\underline{\underline{\Omega}} = \dot{\underline{\underline{Q}}}\,\underline{\underline{Q}}^T$

$\Rightarrow$ rigid body angular velocity between the observers

Skew $\underline{\underline{\Omega}} = -\underline{\underline{\Omega}}^T$ is skew symmetric

$$\frac{d}{dt}\,\underline{\underline{Q}}\,\underline{\underline{Q}}^T = \cancel{I} \; 0 \quad \Rightarrow \text{HW8}$$

Non-objective part of $\underline{\underline{\ell}} = \nabla_x v$ is skew-sym

$\Rightarrow$ take $\mathrm{sym}(\nabla_x v)$ to be objective

$$\Rightarrow \quad \boxed{\underline{\underline{d}} = \mathrm{sym}(\nabla_x \underline{v}) = \frac{1}{2}(\nabla_x \underline{v} + \nabla_x \underline{v}^T)}$$

rate of strain tensor is objective

$\Rightarrow$ used in constitutive laws!

Objective functions

Fields: $\phi(\underline{x}, t)$ scalar

$\underline{v}(\underline{x}, t)$ vector

$\underline{\underline{S}}(\underline{x}, t)$ tensor

field $\rightarrow$ depends on $\underline{x}$

Constitutive functions are not fields but they take fields as input

internal energy: $\underset{\substack{\uparrow \\ \text{output field}}}{u(\underline{x}, t)} = \underset{\substack{\uparrow \\ \text{constitutive function}}}{\hat{u}}(\rho(\underline{x}, t), \theta(\underline{x}, t))$

input fields

heat flow: $\underline{q}(\underline{x}, t) = \hat{q}(\theta(\underline{x}, t))$

$\hat{q} = -\kappa \nabla \theta$

Cauchy stress: $\underline{\underline{s}}(\underline{x}, t) = \hat{\underline{\underline{s}}}(\underline{\underline{d}}(\underline{x}, t))$

Isotropic functions

Functions that are frame invariant are called isotropic.

$\hat{\phi}$ = scalar func. $\quad \hat{\underline{\omega}}$ = vect. func. $\quad \hat{\underline{\underline{\sigma}}}$ = tensor funct.

θ = scalar field $\quad \underline{v}$ = vect. field $\quad \underline{\underline{S}}$ = tensor field

For two frames related by rigid body rotation $\underline{\underline{Q}}$ we have the following isotropic functions:

$$\hat{\phi}(\theta(\underline{x}')) = \hat{\phi}(\theta(\underline{x})) \qquad \hat{\phi}(\underline{\underline{Q}}\,\underline{v}(\underline{x}')) = \hat{\phi}(\underline{v}(\underline{x})) \qquad \hat{\phi}(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \hat{\phi}(\underline{\underline{S}})$$

$$\hat{\underline{\omega}}(\theta(\underline{x}')) = \underline{\underline{Q}}\,\hat{\underline{\omega}}(\theta(\underline{x})) \qquad \hat{\underline{\omega}}(\underbrace{\underline{\underline{Q}}\,\underline{v}}_{\underline{v}'}) = \underline{\underline{Q}}\,\hat{\underline{\omega}}(\underline{v}(\underline{x})) \qquad \hat{\underline{\omega}}(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\hat{\underline{\omega}}(\underline{\underline{S}})$$

$$\hat{\underline{\underline{\sigma}}}(\theta(\underline{x}')) = \underline{\underline{Q}}\,\sigma(\theta(\underline{x}))\underline{\underline{Q}}^T \qquad \hat{\underline{\underline{\sigma}}}(\underline{\underline{Q}}\,\underline{v}) = \underline{\underline{Q}}\,\sigma(\underline{v})\,\underline{\underline{Q}}^T \qquad \hat{\underline{\underline{\sigma}}}(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\hat{\sigma}(\underline{\underline{S}})\,\underline{\underline{Q}}^T$$

Examples:

1) $\hat{\phi}(\underline{\underline{S}}) = \det(\underline{\underline{S}})$

$$\hat{\phi}(\underline{\underline{S}}') = \hat{\phi}(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \det(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) =$$

$$= \det(\underline{\underline{Q}})\,\det(\underline{\underline{S}})\,\det(\underline{\underline{Q}}^T) = \det(\underline{\underline{S}})$$

2) $\hat{\underline{u}}(\underline{v}, \underline{\underline{A}}) = \underline{\underline{A}}\,\underline{v}$

$$\hat{\underline{u}}(\underbrace{\underline{\underline{Q}}\,\underline{v}}_{\underline{v}'}, \underbrace{\underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T}_{\underline{\underline{A}}'}) = \underline{\underline{Q}}\,A\,\underline{\underline{Q}}^T\underline{\underline{Q}}\,\underline{v} = \underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{v} = Q\,\hat{\underline{u}}(\underline{v}, A)$$

Representation theorem for isotropic functions.

Rivlin-Ericksen Rep. Thm

$$\boxed{\underline{\underline{G}}(\underline{\underline{A}}) = \alpha_0(I_A)\underline{\underline{I}} + \alpha_1(I_A)\underline{\underline{A}} + \alpha_2(I_A)\underline{\underline{A}}^2}$$

where $\alpha_0, \alpha_1, \alpha_2$ are scalar functions of the set of invariants of $\underline{\underline{A}}$ $\quad I_A = \{I_1(A), I_2(\underline{\underline{A}}), I_3(\underline{\underline{A}})\}$

- $\underline{\underline{G}}$ is sym if $\underline{\underline{A}}$ is sym.
- $\underline{\underline{G}}$ is isotropic

$$\underline{\underline{G}}(\underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T) = \alpha_0\underline{\underline{I}} + \alpha_1\,\underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T + \alpha_3\,QA\xcancel{Q^T Q}^{\,I}AQ^T$$

$$= \alpha_0\underline{\underline{Q}}\,\underline{\underline{Q}}^T + \alpha_1\,\underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T + \alpha_3\,\underline{\underline{Q}}\,\underline{\underline{A}}^2\underline{\underline{Q}}^T$$

$$= \underline{\underline{Q}}(\alpha_0\underline{\underline{I}} + \alpha_1\underline{\underline{A}}' + \alpha_3\underline{\underline{A}}^2)\,Q^T$$

$$= \underline{\underline{Q}}\,\underline{\underline{G}}(\underline{\underline{A}})\,\underline{\underline{Q}}^T$$

only true if $\alpha_0, \alpha_1, \alpha_3$ depend only on invariants of A