Lecture 24: Representation Thm

Logistics: HW 8 is due

HW 7 next Tu is last opportunity

Last time: Constitutive Theory

$$\boxed{\underline{\underline{G}}(\underline{\underline{A}}) = \lambda \, \mathrm{tr}(\underline{\underline{A}})\underline{\underline{I}} + 2\mu \, \mathrm{sym}(\underline{\underline{A}})}$$

linear elastic: $\underline{\underline{A}} = \nabla \underline{u}$

Newtonian fluid: $\underline{\underline{A}} = \nabla \underline{v}$

- Objectivity / Frame Invariance

  $\{\underline{e}_i\}$ & $\{\underline{e}_i'\}$

  $$\boxed{\begin{aligned} \underline{v} &= \underline{\underline{Q}}\, \underline{v}' \\ \underline{\underline{S}} &= \underline{\underline{Q}}\, \underline{\underline{S}}' \underline{\underline{Q}}^T \end{aligned}}$$

- Change in observer

  $\underline{x}' = \underline{\underline{Q}}(t)\, \underline{\varphi}(\underline{X}, t) + \underline{c}(t)$ Eulerian transf.

  $\Rightarrow \underline{\underline{C}}$ is frame invariant

  $\nabla \underline{v}$ not frame invariant

  $\Rightarrow \mathrm{sym}(\nabla \underline{v})$

- Isotropic functions: $\boxed{\hat{\underline{\underline{G}}}(\underbrace{\underline{\underline{Q}}\, \underline{\underline{S}}\, \underline{\underline{Q}}^T}_{\underline{\underline{S}}'}) = \underbrace{\underline{\underline{Q}}\, \hat{\underline{\underline{G}}}(\underline{\underline{S}})\, \underline{\underline{Q}}^T}_{\hat{\underline{\underline{G}}}'}}$

- Representation Theorem

Rivlin - Ericksen Rep. Thm for isotropic tens. func.

$$\boxed{\underline{\underline{G}}(\underline{\underline{A}}) = \alpha_0(I_A)\,\underline{\underline{I}} + \alpha_1(I_A)\,A + \alpha_2(I_A)\,\underline{\underline{A}}^2}$$

Representation for linear isotropic tens. func.

· $\alpha_2 = 0 \qquad \alpha_1 = c_2 \qquad \alpha_0 = c_0\,tr(\underline{\underline{A}}) + c_1$

$\underline{\underline{G}}(\underline{\underline{0}}) = \underline{\underline{0}} \quad \Rightarrow \quad c_1 = 0$

setting $\quad c_0 = \lambda \qquad c_2 = 2\mu$

$$\boxed{\underline{\underline{G}}(\underline{\underline{A}}) = \lambda\, tr(\underline{\underline{A}})\,\underline{\underline{I}} + 2\mu\ sym(A)}$$

Derive linear isotropic Representation Thm

$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$

fluids: $\underline{\underline{\sigma}}(\underline{x},t) = \hat{\underline{\underline{\sigma}}}(\underline{\underline{d}}(\underline{x},t)) \qquad \underline{\underline{d}} = \underline{\underline{d}}^T \qquad \dot{\underline{\underline{\varepsilon}}}$

solids: $\underline{\underline{\sigma}}(\underline{x},t) = \hat{\underline{\underline{\sigma}}}(\underline{\underline{\varepsilon}}(\underline{x},t)) \qquad \underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}^T$

$\varepsilon = \frac{1}{2}(\nabla u + \nabla u^T)$

both input & output of stress response $= sym(\nabla \underline{u})$
function is sym. tensors

Sym $\Rightarrow$ spectral decomposition

$\underline{\underline{S}} = \underline{\underline{S}}^T \quad \Rightarrow \quad \underline{\underline{S}} = \sum_{i=1}^{3} \lambda_i\, \underbrace{\underline{v}_i \otimes \underline{v}_i}_{P_{v_i}} \qquad (\lambda_i, \underline{v}_i)$

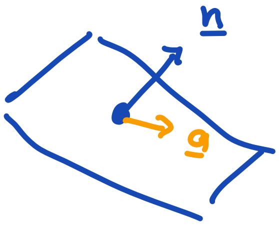
Q: How are $\underline{v}_i$'s of $\underline{\underline{\sigma}}$ and $\underline{\underline{d}}/\underline{\underline{\varepsilon}}$ related?

$\Rightarrow$ same in isotropic material!

$$\underline{\underline{d}}\,\lambda_i = \lambda_i \underline{v}_i \quad \Rightarrow \quad \hat{\underline{\underline{S}}}(\underline{\underline{d}})\,\omega_i = \omega_i \underline{v}_i$$

Can be shown with reflections & projections.

Projection: $\underline{\underline{P}}_n = \underline{n} \otimes \underline{n}$

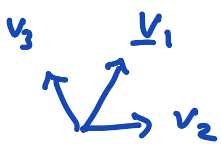


Reflection: $\underline{\underline{R}}_n = \underline{\underline{I}} - 2\underline{n} \otimes \underline{n}$

Note: $\underline{\underline{R}}_n \underline{n} = -\underline{n}$

$\underline{\underline{R}}_n \underline{a} = \underline{a} \Rightarrow \underline{a}$ is in plane

$\underline{a} \cdot \underline{n} = 0$



$\underline{\underline{R}}_{v_1}$ reflection across plane defined by $\underline{v}_1$

$$\underline{\underline{R}}_{v_1} \underline{v}_1 = -\underline{v}_1, \quad \underline{\underline{R}}_{v_1} \underline{v}_2 = \underline{v}_2, \quad \underline{\underline{R}}_{v_1} \underline{v}_3 = \underline{v}_3$$

Step 1: $\boxed{\underline{\underline{R}}_{v_1} \underline{\underline{S}}\, \underline{\underline{R}}_{v_1}^T = \underline{\underline{S}}}$ $\qquad \underline{\underline{S}} = \underline{\underline{S}}^T$

$$= \underline{\underline{R}}_{v_1}\left(\sum_{i=1}^{3} \lambda_i (\underline{v}_i \otimes \underline{v}_i)\right) \underline{\underline{R}}_{v_1}^T$$

$$= \sum_{i=1}^{3} \lambda_i\, \underline{\underline{R}}_{v_1} (\underline{v}_i \otimes \underline{v}_i)\, \underline{\underline{R}}_{v_1}^T$$

use identities: $\underline{\underline{A}}(\underline{a} \otimes \underline{b}) = (\underline{\underline{A}}\underline{a}) \otimes \underline{b}$

$$(\underline{a} \otimes \underline{b})\underline{\underline{B}} = \underline{a} \otimes (\underline{\underline{B}}^T \underline{b})$$

$$\underline{\underline{R}}_{v_1}\,\underline{\underline{S}}\,\underline{\underline{R}}_{v_1}^T = \sum_{i=1}^{3} \lambda_i \,(\underline{\underline{R}}_{v_1}\underline{v}_i) \otimes (\underline{\underline{R}}_{v_1}\underline{v}_i)$$

$$= \lambda_1 (-\underline{v}_1) \otimes (-\underline{v}_1) + \lambda_2\, \underline{v}_2 \otimes \underline{v}_2 + \lambda_3\, \underline{v}_3 \otimes \underline{v}_3$$

$$= \sum_{i=1}^{3} \lambda_i\, \underline{v}_i \otimes \underline{v}_i = \underline{\underline{S}}$$

$$\Rightarrow \quad \underline{\underline{R}}_{v_1}\,\underline{\underline{S}}\,\underline{\underline{R}}_{v_1}^T = \underline{\underline{S}} \quad \Rightarrow$$

Step 2: $\boxed{\underline{\underline{R}}_{v_i}\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{d}}) = \hat{\underline{\underline{\sigma}}}(\underline{\underline{d}})\,\underline{\underline{R}}_{v_i}}$ commute

isotropic function: $\underline{\underline{Q}}\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{d}})\,\underline{\underline{Q}}^T = \hat{\underline{\underline{\sigma}}}(\underline{\underline{Q}}\,\underline{\underline{d}}\,\underline{\underline{Q}}^T)$

$\underline{\underline{Q}}$ = orthogonal (rotation or reflection)

$\underline{\underline{Q}} = \underline{\underline{R}}_{v_1}$

$$\underline{\underline{R}}_{v_1}\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{d}})\,\underline{\underline{R}}_{v_1}^T = \hat{\underline{\underline{\sigma}}}(\underline{\underline{R}}_{v_1}\,\underline{\underline{d}}\,\underline{\underline{R}}_{v_1}^T)$$

$$\underline{\underline{R}}_{v_1}\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{d}})\,\underline{\underline{R}}_{v_1}^T = \hat{\underline{\underline{\sigma}}}(\underline{\underline{d}})$$

$$\underline{\underline{R}}_{v_1}\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{d}})\,\cancelto{\underline{\underline{I}}}{\underline{\underline{R}}_{v_1}^T\underline{\underline{R}}_{v_1}} = \hat{\underline{\underline{\sigma}}}(\underline{\underline{d}})\,\underline{\underline{R}}_{v_1}$$

$$\Rightarrow \quad \underline{\underline{R}}_{v_1}\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{d}}) = \hat{\underline{\underline{\sigma}}}(\underline{\underline{d}})\,\underline{\underline{R}}_{v_1}$$

Step 3: $\hat{\underline{\underline{\sigma}}}(\underline{\underline{d}})\,\underline{v}_i = \omega_i\,\underline{v}_i \qquad \underline{\underline{d}}\,\underline{v}_i = \lambda_i\,\underline{v}_i$

$$\underline{\underline{R}}_{v_i}\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{d}})\,\underline{v}_i = \hat{\underline{\underline{\sigma}}}(\underline{\underline{d}})\,\underbrace{\underline{\underline{R}}_{v_i}\,\underline{v}_i}_{-\underline{v}_i}$$

$$=$$

$$\underline{\underline{P}}_{v_i}\,\underbrace{\hat{\underline{\underline{\sigma}}}(\underline{\underline{d}})\,\underline{v}_i} = -\underbrace{\hat{\underline{\underline{\sigma}}}(\underline{\underline{d}})\,\underline{v}_i}$$

$$\hat{\underline{\underline{\sigma}}}(\underline{\underline{d}})\,\underline{v}_i \parallel \underline{v}_i$$

$\Rightarrow$ principal directions of $\underline{\underline{\sigma}}(\underline{x},t) = \hat{\underline{\underline{\sigma}}}(\underline{\underline{d}}(\underline{x},t))$

same, as $\underline{\underline{d}}(\underline{x},t)$

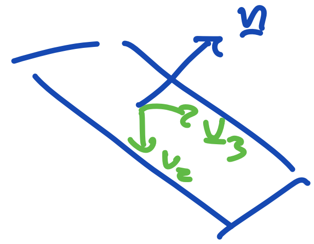
Representation Thm for linear isotropic func.

Note: $\underline{\underline{S}} = \sum_{i=1}^{3} \lambda_i \underbrace{\underline{v}_i \otimes \underline{v}_i}_{\underline{\underline{P}}_{v_i}\ \text{projection tensor}} = \sum_{i=1}^{3} \lambda_i \underline{\underline{P}}_{v_i}$

Eigenproblem: $\underline{\underline{P}}_n \lambda_i = \lambda_i \underline{v}_i$

$\lambda_1 = 1 \qquad \lambda_2 = \lambda_3 = \lambda = 0$

$\underline{v}_1 = \underline{n} \qquad v_2\ \&\ v_3$



any 2 perp. vectors in plane

defined by $\underline{n}$

For an $\underline{\underline{S}}$ with v_1, v_2 $\underline{v}_3$ but λ_1, λ

$$\underline{\underline{S}} = \lambda_1\, \underline{v}_1 \otimes v_1 + \lambda\, \underline{v}_2 \otimes v_2 + \lambda\, \underline{v}_3 \otimes v_3$$

$$= \lambda_1\, \underline{v}_1 \otimes \underline{v}_1 - \lambda\, \underline{v}_1 \otimes \underline{v}_1 + \lambda\, \underline{v}_1 \otimes \underline{v}_1 + \lambda\, \underline{v}_2 \otimes \underline{v}_2 + \lambda\, \underline{v}_3 \otimes \underline{v}_3$$

$$= (\lambda_1 - \lambda)\underbrace{\underline{v}_1 \otimes \underline{v}_1}_{P_{..}} + \lambda \underbrace{\left(\underline{v}_1 \otimes \underline{v}_1 + \underline{v}_2 \otimes \underline{v}_2 + \underline{v}_3 \otimes \underline{v}_3\right)}_{\underline{\underline{I}}}$$

$$\underline{\underline{S}} = \lambda \underline{\underline{I}} + (\lambda_1 - \lambda)\,\underline{\underline{P}}_{v_1}$$

Consider $\underline{\underline{P}}_{\underline{v}_i}$ where $\underline{\underline{d}}\,\underline{v}_i = \lambda \underline{v}_i$

$$\hat{\underline{\underline{\sigma}}}(\underline{\underline{P}}_{v_i})\,\underline{v}_i = \omega_i\,\underline{v}_i$$

$$= \omega\,\underline{\underline{I}} + \underbrace{(\omega_i - \omega)}\,\underline{\underline{P}}_{v_i}$$

$$= \lambda(\underline{v}_1)\,\underline{\underline{I}} + 2\mu(\underline{v}_1)\,\underline{\underline{P}}_{\underline{v}_i}$$

show λ & μ are independent of $\underline{v}_i$

$|\underline{e}| = |\underline{f}| = 1 \qquad \underline{\underline{R}}\,\underline{e} = \underline{f}$

$$\underline{\underline{P}}_f = \underline{f}\otimes\underline{f} = (\underline{\underline{R}}\,\underline{e})\otimes(\underline{\underline{R}}\,\underline{e}) = \underline{\underline{R}}\,(\underline{e}\otimes\underline{e})\,\underline{\underline{R}}^T$$

$$= \underline{\underline{R}}\,\underline{\underline{P}}_e\,\underline{\underline{R}}^T$$

isotropic: $\hat{\underline{\underline{\sigma}}}(\underline{\underline{R}}\,\underline{\underline{P}}_e\,\underline{\underline{R}}^T) = \underline{\underline{R}}\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{P}}_f)\,\underline{\underline{R}}^T$

$$\hat{\underline{\underline{\sigma}}}(\underline{\underline{R}}\,\underline{\underline{P}}_e\,\underline{\underline{R}}^T) = \hat{\underline{\underline{\sigma}}}(\underline{\underline{P}}_e)$$

$$\underline{\underline{R}}\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{P}}_e)\,\underline{\underline{R}} - \hat{\underline{\underline{\sigma}}}(\underline{\underline{P}}_f) = \underline{\underline{0}}$$

$$\hat{\underline{\underline{\sigma}}}(\underline{\underline{P}}_f) = \lambda(\underline{f})\,\underline{\underline{I}} + 2\mu(\underline{f})\,\underline{\underline{P}}_f$$

$$\hat{\underline{\underline{\sigma}}}(\underline{\underline{P}}_e) = \lambda(\underline{e})\,\underline{\underline{I}} + 2\mu(\underline{e})\,P_e$$

$$[\lambda(\underline{e}) - \lambda(\underline{f})]\,\underline{\underline{I}} + 2[\mu(\underline{e}) - \mu(\underline{f})]\,\underline{\underline{P}}_f = \underline{\underline{0}}$$

since $\underline{\underline{I}}$ and $\underline{\underline{P}}_f$ are lin. indep.

$$\Rightarrow \quad \lambda(\underline{e}) = \lambda(\underline{f}) \qquad \mu(\underline{e}) = \mu(\underline{f})$$

λ, μ are const.

$$\Rightarrow \quad \hat{\underline{\underline{\sigma}}}(\underline{\underline{P}}_{v_i}) = \lambda \underline{\underline{I}} + 2\mu \, \underline{\underline{P}}_{v_i}$$

$$\hat{\underline{\underline{\sigma}}}(\underline{\underline{d}}) = \hat{\underline{\underline{\sigma}}}\left(\sum_{i=1}^{3} \omega_i \underline{\underline{P}}_{v_i}\right) = \sum_{i=1}^{3} \omega \, \hat{\underline{\underline{\sigma}}}(\underline{\underline{P}}_{v_i})$$

$$= \sum_{i=1}^{3} \omega_i (\lambda \underline{\underline{I}} + 2\mu \, \underline{\underline{P}}_{v_i})$$

$$\hat{\underline{\underline{\sigma}}}(\underline{\underline{d}}) = \lambda \underbrace{(\omega_1 + \omega_2 + \omega_3)}_{tr(\underline{\underline{d}})} \underline{\underline{I}} + 2\mu \underbrace{(\omega_1 \underline{\underline{P}}_{v_1} + \omega_2 \underline{\underline{P}}_{v_2} + \omega_3 \underline{\underline{P}}_{v_3})}_{\underline{\underline{d}}}$$

$$\Rightarrow \quad \boxed{\hat{\underline{\underline{\sigma}}} = \lambda \, tr(\underline{\underline{d}}) \, \underline{\underline{I}} + 2\mu \, \underline{\underline{d}}}$$

linear elastic constitutive law

$$\underline{\underline{d}} = \underline{\underline{\varepsilon}} = \frac{1}{2}(\nabla \underline{u} + \nabla \underline{u}^T) \qquad tr(\nabla \underline{u}) = tr(sym(\nabla u))$$

$$tr(\nabla u) = \nabla \cdot u$$

$$\hat{\underline{\underline{\sigma}}}(\underline{\underline{\varepsilon}}) = \lambda \, tr(\underline{\underline{\varepsilon}}) \underline{\underline{I}} + 2\mu \underline{\underline{\varepsilon}}$$

$$\underline{\underline{\sigma}}(\underline{u}) = \lambda \nabla \cdot \underline{u} \, \underline{\underline{I}} + \mu (\nabla \underline{u} + \nabla \underline{u}^T)$$

What is missing: Material constraints

Incompressible deformation:

$$\gamma(\underline{\underline{F}}(\underline{X},t)) = 0$$

$$\det(\underline{\underline{F}}) = \frac{dV_x}{dV_X} = 1$$

Incompressibility constraint:

$$\gamma(\underline{\underline{F}}) = \det(\underline{\underline{F}}) - 1 = 0 \qquad \text{Lagrangian } (\underline{X})$$

to get Eulerian incompressibility

$$\dot{\gamma} = \frac{d}{dt}\det(\underline{\underline{F}}) = \underbrace{\det(F)}_{1}(\nabla_x \cdot \underline{v})_m = 0$$

$$\Rightarrow \quad \nabla_x \cdot \underline{v} = 0$$

In presence of constraint:

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^r + \underline{\underline{\sigma}}^a$$

using Lagrangian formalism.

$$\underline{\underline{\sigma}}^r = -p(\underline{x},t)\,\underline{\underline{I}}$$

$$\Rightarrow \quad \underline{\underline{\sigma}} = -p\,\underline{\underline{I}} + \mu(\nabla\underline{v} + \nabla\underline{v}^T)$$