Lecture 25: Isothermal Ideal Fluids

Logistics: - HW 7 complete ✓

- HW 8 2 more HW's

last date is Th 11/30/23

Last time: Representation theorem (linear)

$$\underline{\underline{G}}(\underline{\underline{A}}) = \lambda \operatorname{tr}(A)\, \underline{\underline{I}} + 2\mu \operatorname{sym}(A)$$

elastic: $\underline{\underline{A}} = \nabla \underline{u}$

fluid: $\underline{\underline{A}} = \nabla \underline{v}$

isotropic: stress & strain have same eigenvectors

Form of $\underline{\underline{G}}(\underline{\underline{A}})$ similar to projection:

$$\underline{\underline{S}} = \beta \underline{\underline{I}} + (\beta_1 - \beta)\, \underline{\underline{P}}_{v_1}$$

Spectral decomp: $\quad S = \sum \alpha_i \underbrace{v_i \otimes v_i}_{\underline{\underline{P}}_{v_i}}$

$$\underline{\underline{G}}(\underline{\underline{A}}) = \sum_{i=1}^{3} \alpha_i\, \underline{\underline{G}}(\underline{\underline{P}}_{v_i})$$

Today: Isothermal fluid mechanics

→ Ideal fluids

Isothermal Fluid Mechanics

→ Eulerian balance laws

→ neglect thermal effects

10 equations:

$\underline{v}_m = \dot{\underline{\varphi}}$ 3 kinematic

$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0$ 1 mass balance

$\rho \dot{\underline{v}} - \nabla \cdot \underline{\underline{\sigma}} = \rho \underline{b}$ 3 lin. mom.

$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$ + 3 ang. mom. $(\sigma_{12} = \sigma_{21}, \sigma_{13} = \sigma_{31}, \sigma_{23} = \sigma_{32})$

10

16 unknown quantities

$\underline{\varphi}$ $\underline{v}$ ρ $\underline{\underline{\sigma}}$ $3+3+1+9=16$

⇒ system of eqns is not closed!

Constitutive equation that relates the 6 independent components of $\underline{\underline{\sigma}}$ to $\underline{\varphi}, \underline{v}, \rho$.

If there is a material constraint, we add both equation $\gamma(\underline{F}) = 0$ and an unknown q.

$\gamma = \det(F) - 1 = 0$

Lagrange multiplier

Ideal Fluids

A fluid is ideal if:

1) Uniform reference density $\rho_0(\underline{X}) = \rho_0 > 0$

2) Fluid is incompressible: $\nabla \cdot \underline{v} = 0$

3) Cauchy stress is sperical: $\underline{\underline{\sigma}} = -p\,\underline{\underline{I}}$

$$\underline{\underline{\sigma}} = \mathrm{sph}(\underline{\underline{\sigma}}) + \mathrm{dev}(\underline{\underline{\sigma}})$$

$\Rightarrow$ no shear stresses: $\underline{t} = \underline{\underline{\sigma}}\,\underline{n} = -p\,\underline{n}$

$1 + 2 \quad \Rightarrow \quad \rho(\underline{x}, t) = \rho_0$

Substitute into mass balance:

$$\cancel{\frac{\partial \rho_0}{\partial t}}^{\,0} + \nabla \cdot (\rho_0 \underline{v}) = 0 \quad \Rightarrow \quad \nabla \cdot \underline{v} = 0$$

Substitute into linear momentum balance:

$$\rho_0 \dot{\underline{v}} = \nabla \cdot (-p\,\underline{\underline{I}}) + \rho_0 \underline{b}$$

$$\dot{\underline{v}} = \frac{\partial \underline{v}}{\partial t} + (\nabla \underline{v})\,\underline{v} \quad \text{and} \quad -\nabla \cdot (p(\underline{x})\,\underline{\underline{I}}) = -\nabla p$$

$\Rightarrow$ closed system for $\underline{v}$ and p

$$\boxed{\begin{aligned} \rho_0\left(\frac{\partial \underline{v}}{\partial t} + (\nabla \underline{v})\,\underline{v}\right) &= -\nabla p + \rho_0 \underline{b} \\ \nabla \cdot \underline{v} &= 0 \end{aligned}}$$

Euler Equations

4 equations and 4 unknowns $\underline{v}$ and p

Note: p has undetermined constant.

$\Rightarrow$ Euler equations are non-linear

Stress power in Ideal fluid

$$\underline{\underline{\sigma}} : \underline{\underline{d}} = -p\, \underline{\underline{I}} : \mathrm{sym}(\nabla \underline{v})$$

using: $\underline{\underline{I}} : \underline{\underline{A}} = \mathrm{tr}(\underline{\underline{A}})$ and $\mathrm{tr}(\mathrm{sym}(\underline{\underline{A}})) = \mathrm{tr}(\underline{\underline{A}})$

$$\underline{\underline{\sigma}} : \underline{\underline{d}} = -p\, \mathrm{tr}(\nabla \underline{v}) = -p\, \nabla \cdot \underline{v} = 0$$

$\Rightarrow$ stress power in ideal fluid is zero

Bernoulli Streamline theorem (steady)

from HW: $(\nabla \underline{v})\underline{v} = (\nabla \times \underline{v}) \times \underline{v} + \frac{1}{2} \nabla |\underline{v}|^2$

subst. into Euler Eqns:

$$\frac{\partial \underline{v}}{\partial t} + (\nabla \times \underline{v}) \times \underline{v} = -\frac{1}{2}\nabla |\underline{v}|^2 - \frac{1}{\rho_0}\nabla p + \underline{b}$$

for a conservative body force: $\underline{b} = -\nabla \psi$

$$\frac{\partial \underline{v}}{\partial t} + (\nabla \times \underline{v}) \times \underline{v} = -\nabla\left(\frac{1}{2}|\underline{v}|^2 + \frac{p}{\rho_0} + \psi\right) = -\nabla H$$

$$H = \frac{1}{2}|\underline{v}|^2 + \frac{p}{\rho_0} + \psi$$

where $\psi = gz$ for gravity

H has units of energy/mass

$E_K = \frac{1}{2} m |\underline{v}|^2 \qquad E_G = mgz \qquad E_E = m \int_{p_0}^{p} \frac{dp}{\rho} = m \frac{p - \cancelto{0}{p_0}}{\rho_0}$

$$\boxed{H = \frac{E}{m} = \frac{E_K}{m} + \frac{E_E}{m} + \frac{E_G}{m} = \frac{1}{2}|\underline{v}|^2 + \frac{p}{\rho_0} + gz}$$

Euler Equation: $\frac{\partial \underline{v}}{\partial t} + (\nabla \times \underline{v}) \times \underline{v} = -\nabla H$

Steady Flow: $\frac{\partial \underline{v}}{\partial t} = 0$

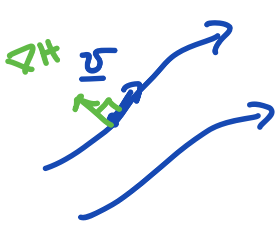
$$(\nabla \times \underline{v}) \times \underline{v} = -\nabla H$$

take dot product with $\underline{v}$

$$\cancelto{0}{\underline{v} \cdot (\nabla \times \underline{v}) \times \underline{v}} = -\underline{v} \cdot \nabla H$$

because $\nabla \times \underline{v} \perp \underline{v}$

$\Rightarrow \quad \boxed{\underline{v} \cdot \nabla H = 0}$ Bernoulli's Thm for steady flow implies that H is constant along a streamline.

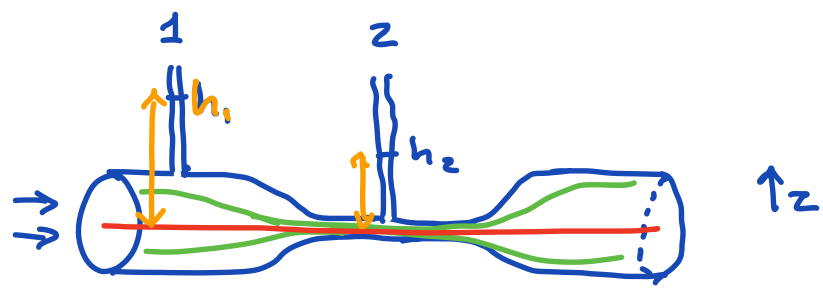


streamline is integral curve of $\underline{v}$

$\Rightarrow$ Energy is conserved in ideal flow $\underline{\underline{e}} : \underline{\underline{d}} = 0$

Examples:

1) <u>Venturi meter</u>



measure flow velocity

Converging-diverging horizontal tube

with 2 manometers

H is const along center streamline

$$H = \frac{p_1}{\rho_0} + \frac{1}{2} v_1^2 = \frac{p_2}{\rho_0} + \frac{1}{2} v_2^2 \qquad (z = 0)$$

mass conservation

$$A_1 v_1 = A_2 v_2 \quad \Rightarrow \quad v_2 = \frac{A_1}{A_2} v_1$$

$$p_1 - p_2 = \frac{\rho_0}{2} \left(v_2^2 - v_1^2 \right) = \frac{\rho_0}{2} \left(\left(\frac{A_1}{A_2} \right)^2 - 1 \right) v_1^2$$

hydrostatic pressure in manometers:

$$p_1 - p_0 = \rho_0 g h_1 \quad \text{and} \quad p_2 - p_0 = \rho_0 g h_2$$

$$\rho_0 g \, (h_1 - h_2) = \frac{\rho_0}{2} \left(\left(\frac{A_1}{A_2} \right)^2 - 1 \right) v_1^2$$

Measure v_1 : $\boxed{v_1^2 = \frac{2g(h_1 - h_2)}{(A_1^2/A_2^2 - 1)}}$

Irrotational motion

A velocity field is irrotational if $\nabla \times \underline{v} = 0$

$\Rightarrow$ material particles experience no net rotation.

Velocity potential:

Helmholtz decomposition:

$$\underline{v} = -\nabla\phi + \nabla \times \underline{\Psi}$$

for irrotational flow

$$\nabla \times \underline{v} = -\cancel{\nabla \times \nabla\phi} + \nabla \times \nabla \times \underline{\Psi} = 0$$

$$\Rightarrow \underline{\Psi} = \underline{0}$$

ϕ is scaler velocity potential for irrotational flow

$$\boxed{\underline{v} = -\nabla\phi}$$

Irrotational + incompressible ($\nabla \cdot \underline{v} = 0$)

$\Rightarrow$ $\boxed{-\nabla^2\phi = 0}$ Laplace Eqn irrotational + steady

$$(\cancel{\nabla \times \underline{v}}^{\,0}) \times \underline{v} = -\nabla H = 0$$

$\Rightarrow$ H is constant throughout fluid.

Time dependent irrotational flows

Start from Euler equation:

$$\frac{\partial \underline{v}}{\partial t} + (\cancel{\nabla \times \underline{v}}^{\,0}) \times \underline{v} = -\tfrac{1}{2}\underline{\nabla}|\underline{v}|^2 - \frac{1}{\rho_0}\underline{\nabla} p + \underline{b}$$

irrotational $\underline{v} = -\nabla\phi$

substitute:

$$\frac{\partial \underline{v}}{\partial t} = \frac{\partial}{\partial t}(-\nabla\phi) = -\underline{\nabla}\frac{\partial \phi}{\partial t}$$

$$\nabla\left(-\frac{\partial\phi}{\partial t} + \frac{1}{2}|\underline{v}|^2 + \frac{p}{\rho_0} - gz\right) = 0$$

this implies

$$\boxed{\begin{aligned} \frac{\partial\phi}{\partial t} + \frac{1}{2}|\underline{v}|^2 + \frac{p}{\rho_0} - gz &= f(t) \\ -\nabla^2\phi &= 0 \end{aligned}}$$

Bernoulli's Thm for irrotational flow

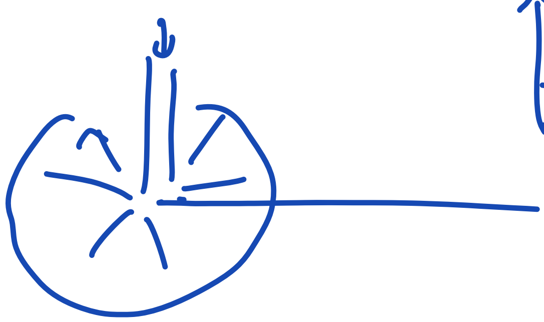
$\Rightarrow$ understand evolution of vorticity

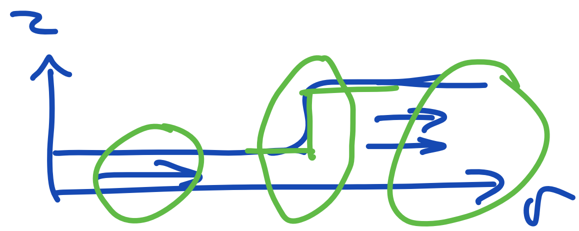
Vorticity equation:

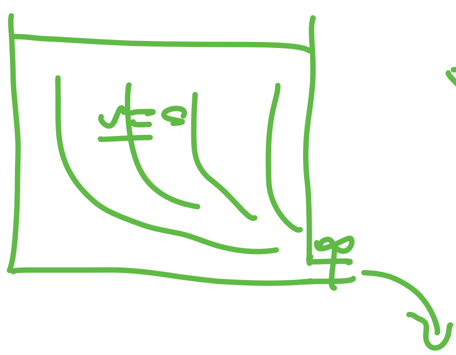
$$\underline{\omega} = \nabla \times \underline{v}$$

$$\boxed{\dot{\underline{\omega}} - (\nabla \underline{v})\underline{\omega} = 0}$$ Vorticity equation

initially irrotational ideal fluid remains irrotational !







$$\nabla \times \underline{v} = \underline{0}$$