Lecture 25: Newtonian Fluids

Logistics: - HW6 is graded

- HW8 please turn it in .

Comment on HW6:

$$\nabla \underline{v} = \frac{\partial}{\partial x_j}\underline{e}_j \otimes v_i \underline{e}_i = \frac{\partial v_i}{\partial x_j}\underline{e}_i \otimes \underline{e}_j$$

$$\nabla \cdot \underline{v} = \frac{\partial}{\partial x_j}\underline{e}_j \cdot v_i \underline{e}_i = \frac{\partial v_i}{\partial x_j}\underbrace{\underline{e}_i \cdot \underline{e}_j}_{\delta_{ij}} = \frac{\partial v_i}{\partial x_i}$$

$$\nabla \cdot \underline{\underline{s}} = s_{ij,j} \qquad \nabla \cdot \underline{v} = \mathrm{tr}(\nabla v$$

$\Rightarrow$ can get you into trouble.

Today $\rightarrow$ Newtonian fluid

After Thanksgivings $\rightarrow$ 1 week

$\Rightarrow$ Stokes fluids $\rightarrow$ glaciers

$\Rightarrow$ creep $\rightarrow$ non-linearity

Newtonian Fluids

A fluid is incompressible Newtonian if:

1) Reference mass density $\rho_0(\underline{X}) = \rho_0$

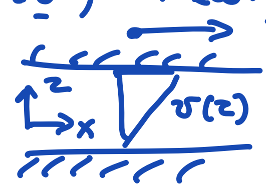
2) Fluid is incompressible $\nabla \cdot \underline{v} = 0$

3) Cauchy stress has form

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^r + \underline{\underline{\sigma}}^a$$

$$\underline{\underline{\sigma}}^r = -p\,\underline{\underline{I}} \qquad p = \text{pressure}$$

$$\underline{\underline{\sigma}}^a = 2\mu\,\mathrm{sym}(\nabla \underline{v}) = \mu(\nabla\underline{v} + \nabla\underline{v}^T) \quad \text{Newtonian}$$



Newton: $\tau \sim \dfrac{dv}{dz}$

$$\tau = \mu \frac{dv}{dz}$$

$\Rightarrow$ both $\underline{\underline{\sigma}}^r$ and $\underline{\underline{\sigma}}^a$ are objective

$\Rightarrow$ $\boxed{\underline{\underline{\sigma}} = -p\,\underline{\underline{I}} + \mu(\nabla\underline{v} + \nabla\underline{v}^T)}$ is also objective

limit of $\mu \to 0$ reduces to ideal fluid.

Navier-Stokes Equations

lin. mom. balance: $\rho \dot{\underline{v}} = \nabla \cdot \underline{\underline{\sigma}} + \rho \underline{b}$

subst: $\rho = \rho_0$ $\quad \underline{\underline{\sigma}} = -p\underline{\underline{I}} + \mu (\nabla \underline{v} + \nabla \underline{v}^T)$

$\nabla \cdot$

$$\rho_0 \dot{\underline{v}} = \nabla \cdot \left(-p\,\underline{\underline{I}} + 2\mu\, \mathrm{sym}(\nabla \underline{v})\right) + \rho_0 \underline{b}$$

mat. deriv: $\dot{\underline{v}} = \frac{\partial v}{\partial t} + (\nabla \underline{v})\, \underline{v}$

assume $\mu = \text{const.}$

$$\nabla \cdot \underline{\underline{\sigma}} = -\nabla p + \mu\, \nabla \cdot \nabla \underline{v} + \mu\, \nabla \cdot \nabla \underline{v}^T$$

$$\nabla \cdot \nabla \underline{v} = \nabla^2 \underline{v}$$

$$\nabla \cdot (\nabla \underline{v})^T = v_{j,ij}\, \underline{e}_i = v_{j,ji}\, \underline{e}_i = \nabla(\cancel{\nabla \cdot \underline{v}})^{\,0}$$

$$\Rightarrow \quad \nabla \cdot \underline{\underline{\sigma}} = -\nabla p + \mu \nabla^2 \underline{v}$$

last lecture

ideal $\mu = 0$

so that:

$$\boxed{\begin{gathered}\rho_0 \left[\frac{\partial v}{\partial t} + (\nabla \underline{v})\, \underline{v}\right] = \cancel{\mu \nabla^2 \underline{v}} - \nabla p + \rho_0 b \\ \nabla \cdot \underline{v} = 0\end{gathered}}$$

Navier Stokes equations

$$(\nabla \underline{v})\, \underline{v} = (\underline{v} \cdot \nabla)\, \underline{v}$$

Energy dissipation

Stress power of Newtonian fluid: $\underline{\underline{d}} = \text{sym}(\nabla \underline{v})$

$$\underline{\underline{\sigma}} : \underline{\underline{d}} = (-p\underline{\underline{I}} + 2\mu \underline{\underline{d}}) : \underline{\underline{d}}$$

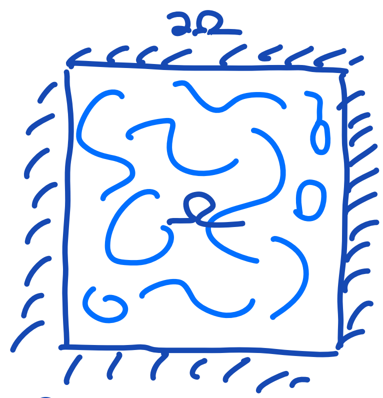
$$= -p \underbrace{\underline{\underline{I}} : \underline{\underline{d}}}_{\nabla \cdot \underline{v} = 0} + 2\mu \, \underline{\underline{d}} : \underline{\underline{d}}$$

$$\underline{\underline{\sigma}} : \underline{\underline{d}} = 2\mu \underbrace{\underline{\underline{d}} : \underline{\underline{d}}}_{>0}$$

if $\mu > 0$ then energy is dissipated by the flow

Kinetic Energy in Fluid Motion

Dissipation of kinetic energy in ideal & Newtonian fluids:



Show how energy decays in closed domain with initial flow.

First some results:

1) Integration by parts in fixed domain Ω
with "no slip" boundaries $\underline{v} = \underline{0}$ on $\partial\Omega$

$$\boxed{\int_\Omega (\nabla^2 \underline{v}) \cdot \underline{v} \, dV = - \int_\Omega \nabla \underline{v} : \nabla \underline{v} \, dV}$$

$$(v_{i,j} v_i)_{,j} = v_{i,jj} v_i + v_{i,j} v_{i,j}$$

$$(\nabla^2 \underline{v}) \cdot \underline{v} = v_{i,jj} v_i = (v_{i,j} v_i)_{,j} - v_{i,j} v_{i,j}$$
$$= \nabla \cdot ((\nabla \underline{v})^T \underline{v}) - \nabla \underline{v} : \nabla \underline{v}$$

substitute:

$$\int_\Omega (\nabla^2 \underline{v}) \cdot \underline{v} \, dV = \int_{\partial\Omega} (\nabla v)^T \cancel{\underline{v}} \cdot \underline{n} \, dA - \int_\Omega \nabla \underline{v} : \nabla \underline{v} \, dV$$

2) Poincare Inequality

$$\| \underline{u} \|_\Omega \leq \lambda \| \nabla \underline{u} \|_\Omega \qquad \text{for all } \underline{u} \neq 0 \qquad \lambda > 0$$

$\underline{u} \cdot \underline{u}$ $\qquad$ $\nabla u : \nabla u$

$$\boxed{\int_\Omega |\underline{u}|^2 \, dV \leq \lambda \int_\Omega \nabla \underline{u} : \nabla \underline{u} \, dV}$$

$\#$ $\qquad$ L^2 $\qquad$ $\frac{1}{L}\#$ $\quad$ $\frac{1}{L}\#$

λ has units of length square $\Rightarrow$ scales with area

By definition of Kinetic energy

$$\frac{d}{dt}K(t) = \frac{d}{dt}\int_\Omega \frac{1}{2}\rho_0 |\underline{v}|^2 d\Omega = \int_\Omega \frac{1}{2}\rho_0 \frac{d}{dt}|\underline{v}|^2 dV$$

$$= \int \rho_0 \dot{\underline{v}}\cdot\underline{v}\, dV$$

from N-S: $\rho\dot{\underline{v}} = \mu\nabla^2\underline{v} - \rho\nabla\psi$ $\qquad \underline{b} = -\nabla\psi$

$$\frac{d}{dt}K(t) = \int_\Omega (\mu\nabla^2\underline{v} - \rho\nabla\psi)\cdot\underline{v}\, dV$$

$$= \int_\Omega (\mu\nabla^2\underline{v})\cdot\underline{v}\, dV - \int_\Omega \rho\, \cancel{\nabla\psi\cdot\underline{v}}\, dV \quad 0$$

$$\nabla\cdot(\psi\underline{v}) = \nabla\psi\cdot\underline{v} + (\cancel{\nabla\cdot\underline{v}})\psi \quad 0$$

$$\nabla\cdot(\psi\underline{v}) = \nabla\psi\cdot\underline{v}$$

$$\int_\Omega \nabla\psi\cdot\underline{v}\, dV = \int_\Omega \nabla\cdot(\psi\underline{v})\, dV = \oint_{\partial\Omega}\psi\,\cancel{\underline{v}}\cdot n\, dA \quad 0$$

$$\Rightarrow \frac{d}{dt}K(t) = \mu\int_\Omega \nabla^2\underline{v}\cdot\underline{v}\, dV = -\mu\int_\Omega \underbrace{\nabla\underline{v}:\nabla\underline{v}}_{\le \frac{1}{\lambda}\int |\underline{v}|^2 dV}\, dV$$

$$\frac{d}{dt}K(t) \le -\frac{\mu}{\lambda}\int_\Omega |\underline{v}|^2 dV = -\frac{2\mu}{\lambda\rho_0}K(t)$$

ODE for K

$$\boxed{\frac{d}{dt} K(t) \leq -\frac{2\mu}{\lambda\rho_0} K(t)}$$

λ = depends on domain

Solve by separation of variables

$$\frac{dK}{K} \leq -\underbrace{\frac{2\mu}{\rho_0\lambda}}_{\alpha} dt = -\alpha\, dt$$

$$\ln(K) \leq -\alpha t + c_0$$

$$K \leq c_1 e^{-\alpha t}$$

Initial condition: $K(0) \leq c = K_0$

$$\Rightarrow \quad \boxed{K(t) \leq K_0\, e^{-\frac{2\mu}{\lambda\rho_0} t}}$$

In absence of fluid motion on boundary the kinetic energy decays exponentially

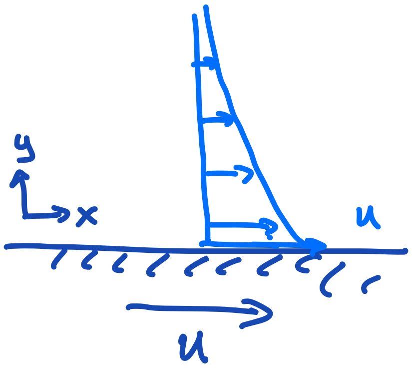
Rate of decay: $\boxed{\nu = \frac{\mu}{\rho_0}}$ Kinematic viscosity

$\nu \quad \left[\frac{L^2}{T}\right] \Rightarrow$ diffusion

μ = dynamic viscosity

Rayleighs first problem

- Semi-infinite half space
- Fluid is initially stationary
- Impulsively started plate with velocity U



$$\rho_0 \frac{\partial \underline{v}}{\partial t} + (\nabla \underline{v})\,\underline{v} = \mu \nabla^2 \underline{v} - \nabla p + \rho g$$

reduced pressure:

$$-\nabla p + \rho g = -\nabla p - \rho g \hat{\underline{y}} = -\nabla(\underbrace{p+\rho g y}_{\pi}) = -\nabla \pi$$

$\pi = p + \rho g z$ = reduced pressure

$(\nabla v)v$ is quadratic in $\underline{v}$

if $\underline{v}$ small $(\nabla v)\,v \rightarrow$ is negligible.

$$\Rightarrow \boxed{\rho \frac{\partial \underline{v}}{\partial t} = \mu \nabla^2 \underline{v} - \nabla \pi}$$

const

Stokes

is linear

$\underline{v} = \begin{pmatrix} u \\ w \end{pmatrix} \begin{matrix} x \\ y \end{matrix}$

Simplify equs:

because domain is infinite but $|\pi| < \infty \Rightarrow \frac{\partial \pi}{\partial x} = 0$

flow is horizontal : $\underline{v} = \begin{pmatrix} u \\ 0 \end{pmatrix}$ $\qquad w = 0$

from continuity: $\nabla \cdot v = \frac{\partial u}{\partial x} + \cancel{\frac{\partial w}{\partial y}}^{\,0} = 0$

$\Rightarrow \quad \underline{u = u(y)}$

$\nabla^2 \underline{v} = v_{i,jj}\, e_i \qquad i,j \in \{1,2\}$

$v_1 = u \qquad v_2 = w$

$$\nabla^2 \underline{v} = \begin{pmatrix} v_{1,11} & v_{1,22} \\ v_{2,11} & v_{2,22} \end{pmatrix} = \begin{pmatrix} \cancel{u_{xx}}^{\,0} & u_{yy} \\ \cancel{w_{xx}}^{\,0} & \cancel{w_{yy}}^{\,0} \end{pmatrix} = \begin{pmatrix} \underline{u_{yy}} \\ 0 \end{pmatrix}$$

substitute

x-mom: $\rho_0 \frac{\partial u}{\partial t} - \mu \frac{\partial^2 u}{\partial y^2} = 0$

y-mom: $\qquad 0 = -\frac{\partial \Pi}{\partial y}$

$$\Rightarrow \qquad \boxed{\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}} \qquad u = u(y)$$

BC: $u(t, y=0) = U$

IC: $u(t=0, y) = 0$

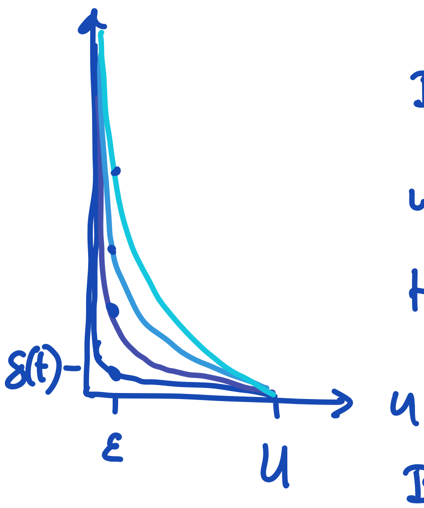
$\Rightarrow$ diffusion equation

Heat equation: $\cancel{\rho c_p} \frac{\partial T}{\partial t} = \frac{k}{\rho c_p} \nabla^2 T = \underbrace{\frac{K}{\rho c_p}}_{\alpha} \frac{\partial^2 T}{\partial x^2}$

Classic analytic solution for diffusion:

$$\Rightarrow \boxed{u(y,t) = U\left(1-\mathrm{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)\right)}$$

⇒ see notes for self-similar ansatz.



Diffusive boundary layer where momentum is added to fluid from moving boundary

Boundary layer thickness:

$$\boxed{\delta \sim 2\sqrt{\nu t}}$$

10^{16}

$$\frac{\mu}{\rho} = \frac{10^{13}}{10^{3}} \ \mathrm{Pa\,s}$$

$\nu = \frac{\mu}{\rho_0}$ is momentum diffusivity