Lecture 26: Creeping Flow - Stokes Eqn.

Logistics: - HW7 graded

Q2: Rigid rotation about fixed point $\underline{y}$

$$\varphi = \underline{y} + \underline{Q}(\underline{X} - \underline{y})$$

$$\underline{\underline{F}} = \nabla \varphi = \frac{\partial \varphi_i}{\partial X_j}\, \underline{e}_i \otimes \underline{e}_j = \underline{\underline{Q}}$$

$$\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}} = \underline{\underline{Q}}^T \underline{\underline{Q}} = \underline{\underline{I}}$$

- HW8 Thursday (11/30) last chance!
- Course Evaluations

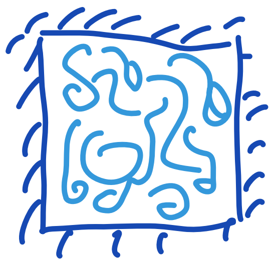
Last time: - Navier-Stokes Equations

Newtonian Stress: $\underline{\underline{\sigma}} = -p\underline{\underline{I}} + \eta(\nabla \underline{v} + \nabla \underline{v}^T)$

$$\boxed{\rho_0\left[\frac{\partial \underline{v}}{\partial t} + (\nabla \underline{v})\underline{v}\right] = \eta \nabla^2 \underline{v} - \nabla p + \rho_0 \underline{g}}$$

- Decay of kinetic Energy

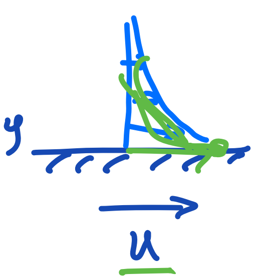
$$\boxed{k(t) \le k_0\, e^{-\frac{2\eta}{\lambda \rho_0} t}}$$



- Rayleigh's problem

$$u(y,t) = U\left(1 - \mathrm{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)\right)$$

$\frac{L^2}{T}$ $\quad \nu = \frac{\eta}{\rho_0}$ mom. diffusivity



Today: - Creeping flows

Scaling of N-S equations

$$\rho \frac{\partial \underline{v}}{\partial t} + (\nabla \underline{v})\,\underline{v} = \eta \nabla^2 \underline{v} - \nabla p + \rho \underline{g}$$

introduce reduced pressure:

$$-\nabla p + \rho \underbrace{\underline{g}}_{-g\hat{\underline{z}}} = -\left(\nabla p + \rho g \hat{\underline{z}}\right) = -\nabla(\underbrace{p + \rho g z}_{\pi}) = -\nabla \pi$$

so we have

$$\rho\left[\frac{\partial \underline{v}}{\partial t} + (\nabla \underline{v})\,\underline{v}\right] = \eta \nabla^2 \underline{v} - \nabla \pi$$

$$\nabla \cdot \underline{v} = 0$$

Non-dimensionalize with generic quantities to define standard dimensionless parameters:

- Dependent variable: $\underline{v}$, π
- Independent variables: $\underline{x}$, t
- Parameters: $\rho \left[\frac{M}{L^3}\right]$ $\eta \left[\frac{M}{LT}\right] \rightarrow \nu = \frac{\eta}{\rho} \left[\frac{L^2}{T}\right]$

  + Geometry, BC, IC

Use parameter combinations to scale variable:

$$\underline{v}' = \frac{\underline{v}}{V_c} \qquad \pi' = \frac{\pi}{\pi_c} \qquad \underline{x}' = \frac{\underline{x}}{x_c} \qquad t' = \frac{t}{t_c}$$

$\Rightarrow$ substitute into governing equations

$$\rho \frac{\partial v}{\partial t} = \rho \frac{\partial(v_c v')}{\partial(t_c t')} = \frac{\rho v_c}{t_c}\frac{\partial v'}{\partial t'}$$

$$\underline{\frac{\rho v_c}{t_c}} \frac{\partial v'}{\partial t'} + \underline{\frac{\rho v_c^2}{x_c}} (\nabla' v') v' = \underline{\frac{\eta v_c}{x_c^2}} \nabla'^2 v' - \underline{\frac{\pi_c}{x_c}} \nabla' \pi'$$

Option 1: Scale to accumulation term

$$1\ \frac{\partial v'}{\partial t'} + \underbrace{\frac{v_c t_c}{x_c}}_{\Pi_1} \frac{t_c}{t_A} (\nabla' v') v' = \underbrace{\frac{\nu t_c}{x_c^2}}_{\Pi_2 = 1} \nabla'^2 v' - \underbrace{\frac{\pi_c t_c}{x_c \rho v_c}}_{\Pi_3 = 1} \nabla' \pi'$$

$\nu = \frac{\eta}{\rho}$ internal

Three dimensionless groups: ⇒ define time scales

$\Pi_1 = \frac{v_c t_c}{x_c} = 1$ ⇒ advective time scale: $t_c = t_A = \frac{x_c}{v_c}$

$\Pi_2 = \frac{\nu t_c}{x_c^2} = 1$ ⇒ diffusive time scale: $\underline{t_c = t_D} = \frac{x_c^2}{\nu}$

Use Π_3 to define pressure scale

$\Pi_3 = \frac{\pi_c t_c}{x_c \rho v_c} = 1$ ⇒ $\pi_c = \frac{x_c \rho v_c}{t_c}$

Choose a diffusive time scale: $t_c = t_D = \frac{x_c^2}{\nu}$

$$\Rightarrow \frac{\partial v'}{\partial t'} + \underbrace{\frac{t_D}{t_A}}_{\Pi_1 = \frac{v_c x_c}{\nu}} (\nabla' v') v' = \nabla'^2 v' - \nabla' \pi'$$

$\Rightarrow$ only one remaining dim. less group.

Reynolds number: $\boxed{Re = \frac{v_c x_c}{\nu} = \frac{t_D}{t_A}}$

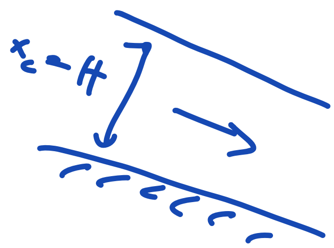
$\Rightarrow$ Peclet number for lin. momentum

Hence we have dropping the primes

$$\frac{\partial \underline{v}}{\partial t} + Re\, (\nabla \underline{v})\, \underline{v} = \nabla^2 \underline{v} - \nabla \pi$$

as $Re \to 0$ advective momentum transfer vanishes

Re for a glacier



$\rho \sim 10^3 \frac{kg}{m^3}$ $\quad$ $v_c \sim 10^2 \frac{m}{yr} \sim 10^{-6} \frac{m}{s}$

$\eta \sim 10^{14}$ Pa s $\quad$ $x_c \sim 10^2$ m

$$Re = \frac{v_c x_c \rho}{\eta} = \frac{10^{-6+2+3}}{10^{14}} = 10^{-1-14} = 10^{-15} \ll 1$$

$$\Rightarrow \frac{\partial \underline{v}}{\partial t} = \nabla^2 \underline{v} - \nabla \pi$$ linear, transient

Is it worth resolving the transient?

$$t_D = \frac{x_c^2 \rho}{\eta} = 10^{4+3-14}\, s = \underline{10^{-7}\, s}$$

much smaller than glacial time scales

Option 2: Scale to mom. diffusion term

$$\rho\frac{v_c}{t_c}\frac{\partial \underline{v}'}{\partial t'} + \rho\frac{v_c^2}{x_c}(\nabla'\underline{v}')\underline{v}' = \eta\frac{v_c}{x_c^2}\nabla'^2\underline{v}' - \frac{\pi_c}{x_c}\nabla'\pi'$$

divide by $\frac{\eta v_c}{x_c^2}$

$$\underset{\substack{\parallel \\ \frac{\eta}{\rho}}}{\frac{x_c^2}{\nu t_c}}\frac{\partial \underline{v}'}{\partial t'} + \frac{v_c x_c}{\nu}(\nabla'\underline{v}')\underline{v}' = \nabla'^2\underline{v}' - \frac{\pi_c x_c}{\eta v_c}\nabla'\pi'$$

$\hookrightarrow \pi_c = \frac{\eta v_c}{x_c}$

choose a time scale: $t_c = t_A = \frac{x_c}{v_c}$

$$\underset{\substack{\uparrow \\ \frac{v_c x_c}{\nu} \sim 10^{-15}}}{Re}\left(\frac{\partial \underline{v}'}{\partial t'} + (\nabla'\underline{v}')\underline{v}\right) = \nabla'^2\underline{v}' - \nabla'\pi'$$

$\Rightarrow$ Stokes Equation

$$\boxed{\begin{aligned} \nabla'^2\underline{v}' &= \nabla\pi \\ \nabla'\cdot\underline{v}' &= 0 \end{aligned}}$$

dimension less

$\mu = \eta$

Redimensionalize: $\underline{v}' = \frac{v}{v_c}$ $\quad \pi'' = \frac{\pi}{\eta\frac{v_c}{x_c}}$ $\quad \underline{x}' = \frac{x}{x_c}$

$$\frac{\cancel{x_c^2}}{\cancel{v_c}}\nabla^2\underline{v} = \frac{\cancel{x_c^2}}{\eta\cancel{v_c}}\nabla\pi$$

$$\boxed{\begin{aligned} \eta\nabla^2\underline{v} &= \nabla\pi \\ \nabla\cdot\underline{v} &= 0 \end{aligned}}$$

dimensional Stokes equation

This is assuming $\eta = \text{const.}$

Properties of Stokes:

1) Linearity

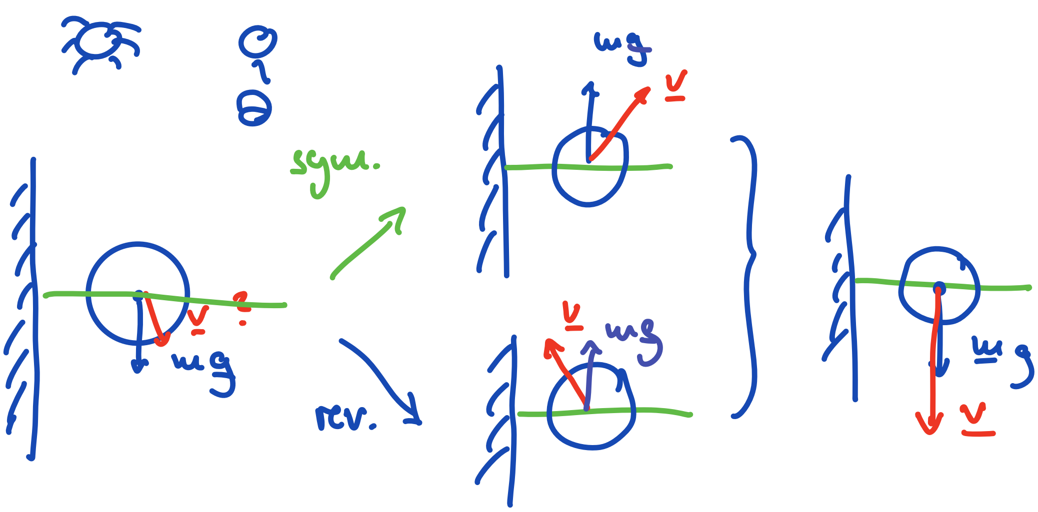
→ Construct solutions by lin. superposition

2) Instantaneous

no time dependence other than that introduced by boundary conditions.

3) Reversibility

If body force and velocity on boundary are reversed so is the velocity everywhere.



In Earth Science most creeping flows are variable viscosity!

$$\rho \frac{\partial \underline{v}}{\partial t} + (\nabla \underline{v})\underline{v} = \nabla \cdot \underline{\underline{\sigma}} + \rho \underline{g}$$

$$\underline{\underline{\sigma}} = -p\underline{\underline{I}} + \eta\left(\nabla \underline{v} + \nabla \underline{v}^T\right)$$

$$\nabla \cdot \underline{\underline{\sigma}} = -\nabla p + \nabla \cdot \left[\eta\left(\nabla \underline{v} + \nabla \underline{v}^T\right)\right]$$

$$\rho \frac{\partial \underline{v}}{\partial t} + (\nabla \underline{v})\underline{v} = \underbrace{\nabla \cdot \left[\eta\left(\nabla \underline{v} + \nabla \underline{v}^T\right)\right]}_{\eta \nabla^2 \underline{v}} - \nabla \pi$$

same scaling as option 2:

$\eta \neq$ const.

$$\eta' = \frac{\eta}{\eta_c}$$

$$Re\left(\cancel{\frac{\partial \underline{v}'}{\partial t'} + (\nabla \underline{v}')\underline{v}'}\right)^{\to 0} = \nabla' \cdot \left[\eta'\left(\nabla' \underline{v}' + \nabla'^T \underline{v}'\right)\right] - \nabla' \pi'$$

$Re \sim 10^{-15}$

Variable viscosity Stokes equation:

$$\boxed{\begin{aligned} \nabla \cdot \left[\eta\left(\nabla \underline{v} + \nabla \underline{v}^T\right)\right] &= \nabla \pi \\ \nabla \cdot \underline{v} &= 0 \end{aligned}}$$

Two common sources of viscosity variation:

1) $\eta = \eta(T)$ temperature dependence

2) $\eta = \eta(\underline{\tau})$ non-Newtonian fluids

→ powerlaw creep