Lecture 28: Power-law creep

Logistics: - HW8 ✓

- Grades are due Dec 14
- Will post final grade Monday Dec 4
- Any changes by Thursday Dec 7
- Will submit grades Friday Dec 8

Last time: - Stokes flow

scaling to diff. mom. transport term

Reynolds #

$$Re\left[\frac{\partial \underline{v}}{\partial t} + (\nabla \underline{v})\underline{v}\right] = \nabla^2 \underline{v} - \nabla \pi$$

creeping flow: $Re \ll 1$ 10^{-15}

$$\boxed{\begin{aligned} \eta(T)\nabla^2 \underline{v} &= \nabla \pi \\ \nabla \cdot \underline{v} &= \underline{0} \end{aligned}}$$

Stokes Equation

(constant viscosity)

- Earth Science: $\eta = \eta(T, \underline{v})$

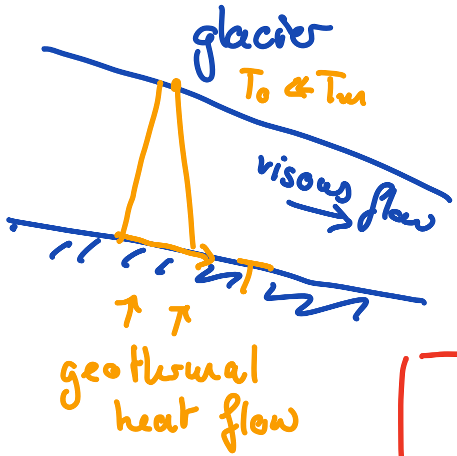
$$\boxed{\begin{aligned} \nabla \cdot \left[\eta(\underline{v})(\nabla \underline{v} + \nabla \underline{v}^T)\right] &= \nabla \pi \\ \nabla \cdot \underline{v} &= \underline{0} \end{aligned}}$$

Today: - Variable viscosity Stokes

1) Temperature dependent: $\eta = \eta(T)$

2) Power-law creep: $\eta = \eta(\underline{v})$ non-lin.

Temperature dependent viscosity



Solid state creep allows a solid to deform like a liquid.

Diffusion creep:

$\Rightarrow$ Newtonian rheology: $\underline{\underline{\sigma}} \sim \eta_T \, \underline{\underline{d}}$ $\cancel{(\underline{d})}^2$

$$\boxed{\eta(T) = \frac{R T d^2}{42 V_m D_{0,V}} \exp\left(\frac{E_A}{RT}\right)}$$

Parameters: d = grain diameter ~ 1 mm

T = temp.

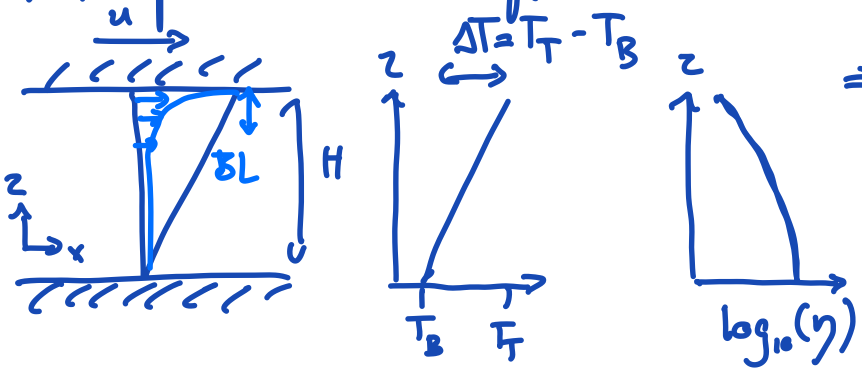
diffusion of vacancies in crystal lattice	V_m = molar volume	$1.97 \cdot 10^{-5}$	$\frac{m^3}{mol}$
	$D_{0,V}$ = vol. diff. constant	$9.1 \cdot 10^{-4}$	$\frac{m^2}{s}$
	E_A = vol. diff. activation energy	59.4	$\frac{kJ}{mol}$
	R = mol. gas const.	8.314	$\frac{J}{K\,mol}$

Arrhenius dependence of T

Often simplified to:

$$\boxed{\eta = \eta_0 \exp\left(\frac{E_A}{RT}\right)} \qquad \eta_0 = \frac{R T_m d^2}{42 V_m D_{0,V}}$$

Example: Couette flow with T-gradient



⇒ formation of low-η boundary layer where shear is localized.

In absence of viscous heating: $\underline{\underline{\sigma}} : \underline{\underline{d}} \approx 0$

⇒ T-field is independent of velocity

→ one-way coupling : $\underline{v} = \underline{v}(T) \quad T \neq T(\underline{v})$

Energy balance eqn:

$$\rho c_p \cancelto{0}{\frac{\partial T}{\partial t}} + \nabla \cdot \left[\underline{v} \rho c_p T - \underset{\substack{\uparrow \\ k(T)}}{k} \nabla T \right] = \cancelto{0}{\underline{\underline{\sigma}} : \underline{\underline{d}}} + \cancelto{0}{\rho \underline{r}}$$

Assumptions:

1) Neglect: r.h.s. = 0

2) Steady state: $\frac{\partial T}{\partial t} = 0$

3) Physical param. are constant: ρ, c_p, k

$$\Rightarrow \quad \nabla \cdot \left[\cancelto{0}{\underline{v}} T - \alpha \nabla T \right] = 0 \qquad \alpha = \frac{k}{\rho c_p}$$

4) Flow is incompressible: $\nabla \cdot \underline{v} = 0$

5 Flow is horizontal $\underline{v} = \begin{bmatrix} v(z) \\ 0 \\ 0 \end{bmatrix} \Rightarrow \nabla T = \begin{bmatrix} 0 \\ 0 \\ \frac{\partial T}{\partial z} \end{bmatrix}$

$$\nabla \cdot [v T] = T \, \cancelto{0}{\nabla \cdot \underline{v}} + \cancelto{0}{\underline{v} \cdot \nabla T}$$

$$\Rightarrow \quad \alpha \nabla^2 T = 0 \qquad T = T(z) \Rightarrow \frac{d^2 T}{dz^2} = 0$$

BC: $T(0) = T_B \qquad T(H) = T_T$

integrating twice: $\boxed{T = T_B + \frac{\Delta T}{H} z}$

Mass & mom. cons:

$$-\nabla \cdot [\underbrace{\eta(T)\,(\nabla \underline{v} + \nabla \underline{v}^T)}_{\text{dev. stress.}}] + \nabla \pi = 0 \qquad \pi = p + \rho g z$$

$$\nabla \cdot \underline{v} = 0$$

deviatoric stress in 2D:

$$\underline{\underline{\tau}} = \eta\,(\nabla \underline{v} + \nabla \underline{v}^T) = \eta \begin{bmatrix} 2\cancelto{0}{v_{x,x}} & v_{x,z} + \cancelto{0}{v_{z,x}} \\ v_{x,z} + \cancelto{0}{v_{z,x}} & 2\cancelto{0}{v_{z,z}} \end{bmatrix}$$

$$\underline{v} = \begin{bmatrix} v_x \\ v_z \end{bmatrix} = \begin{bmatrix} v_x(z) \\ 0 \end{bmatrix} \Rightarrow \quad v_{x,z} \neq 0 \quad v_{x,x} = 0$$

$$\nabla \pi = \begin{pmatrix} \pi_{,x} \\ \cancelto{0}{\pi_{,z}} \end{pmatrix} = \begin{pmatrix} \pi_{,x} \\ 0 \end{pmatrix}$$

all terms in z-mom. balance vanish

$$\Rightarrow \quad \boxed{-\frac{\partial}{\partial z}\left(\eta(T)\,\frac{\partial v}{\partial z}\right) + \frac{\partial \pi}{\partial x} = 0} \qquad v \equiv v_x$$

Couette flow is infinite in x-dir.

$$\Rightarrow \quad \frac{\partial \Pi}{\partial x} = 0$$

Following ODE:
BC
Const:

$$\boxed{\begin{aligned} &\frac{\partial}{\partial z}\left(\eta(T(z))\,\frac{\partial v}{\partial z}\right) = 0 \\ &v(0) = 0, \quad v(H) = u \\ &\eta = \eta_0 \exp\left(\frac{E_a}{RT}\right) \\ &T = T_B + \frac{\Delta T}{H} z \\ &\Delta T = T_T - T_B > 0 \end{aligned}}$$

Integrate once:

$$\eta \frac{\partial v}{\partial z} = c_1 = \tau = \text{shear stress}$$

$$\tau = \eta \frac{\partial v}{\partial z} \qquad \text{definition of } \eta$$

Integrate once more:

$$\frac{\partial v}{\partial z} = \frac{\tau}{\eta(z)}$$

$$v(z) = \tau \int_0^z \frac{dz}{\eta(T(z))}$$

subst:

$$\boxed{v(z) = \tau \int_0^z \frac{dz}{\mu_0 \exp\left(\frac{E_a/R}{T_B + \frac{\Delta T}{H} z}\right)}}$$

There is no closed form solution for this integral but various approximations for ΔT small.

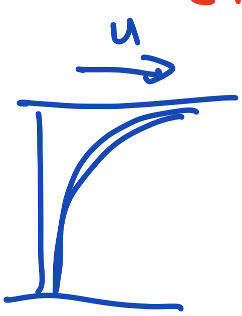
⇒ see notes

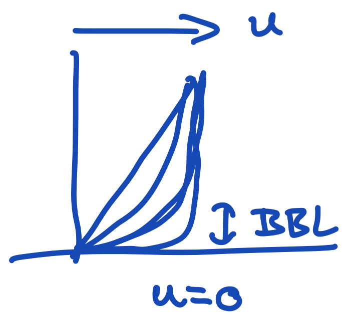
Approx. of velocity profile: $\quad z' = \frac{z}{H}$

$$\boxed{v' = \frac{v(z')}{u} = \frac{\exp\left(\frac{E_a \Delta T}{R T_B^2} \frac{z}{H}\right) - 1}{\exp\left(\frac{E_a \Delta T}{R T_B^2}\right) - 1}}$$

$$a = \frac{E_a}{R\,T_B} \qquad b = \frac{\Delta T}{T_B} \qquad a \cdot b = \frac{E \Delta T}{R T_B^2}$$

$$\Rightarrow \quad \exp(a \cdot b\, z')$$





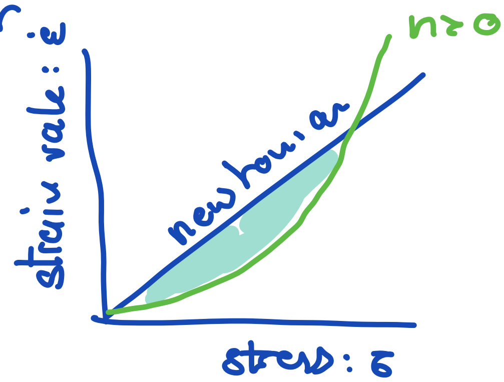
Power law creep

Newtonian fluid: $\underline{\underline{\sigma}} = -p\underline{\underline{I}} + \eta\,(\underbrace{\nabla v + \nabla v^T}_{\dot{\underline{\underline{\varepsilon}}}})$

→ $\eta \neq \eta(\underline{v})$ this is linear

$\dot{\underline{\underline{\varepsilon}}} = \underline{\underline{d}}$ = strain rate tensor

$\boxed{\dot{\varepsilon} = A\,\sigma^n}$



n = stress exponent

A = prefactor

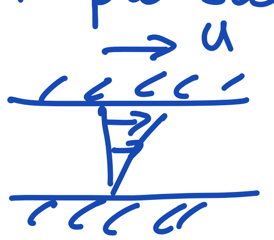
⇒ low stress material experiences less strain

⇒ high stress " " more strain

⇒ common in polycrystalline solids during ductile deformation

⇒ "Rheology of the Earth" good book.

Simple shear



$\underline{\underline{\sigma}} = \begin{bmatrix} 0 & \sigma_s & 0 \\ \sigma_s & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad \dot{\underline{\underline{\varepsilon}}} = \underline{\underline{d}} = \begin{bmatrix} 0 & \dot{\varepsilon}_s & 0 \\ \dot{\varepsilon}_s & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

⇒ for simple geometries $\underline{\underline{\sigma}}$ and $\dot{\underline{\underline{\varepsilon}}}$ have one non-zero entry

Suppose experiments lead:

$$\dot{\underline{\underline{\varepsilon}}}_s = A\, \underline{\underline{\sigma}}_s^{\,n}$$

A is function of p, T & material param.
but 'n' is constant (for same def. mech.)

How do we extend experimental result to general constitutive law in tensor form?

1) Experiments are not affected by confining pressure
$\Rightarrow$ use deviatoric stress

$$\underline{\underline{\sigma}} = -p\,\underline{\underline{I}} + \underline{\underline{\tau}}$$

2) Frame invariant $\Rightarrow$ invariants of $\dot{\underline{\underline{\varepsilon}}}$

$\Rightarrow$ past notes