Lecture 3: Isostacy

Logistics: Office hours Mon 12-1 pm Tue 4-5 pm

Wed 1-2 pm

Last time: - Center of volume & mass

$$\underline{x}_v = \frac{1}{V}\int_B \underline{x}\, dV, \quad \underline{x}_m = \frac{1}{M}\int_B \rho \underline{x}\, dV$$

- review momentum

$$\underline{L} = m\,\underline{v} \quad \& \quad \underline{j} = (\underline{x} - \underline{z}) \times \underline{L}$$

- force & torque

$$\underline{f} = \frac{d\underline{L}}{dt} = m\,\underline{a} \quad \& \quad \underline{\tau} = \frac{d\underline{j}}{dt} = (\underline{x} - \underline{z}) \times \underline{f}$$

- body forces & surface forces

  Examples: Weight & Buoyancy

- traction $\underline{t}$ $\quad (\underline{t} = -p\,\hat{\underline{n}})$

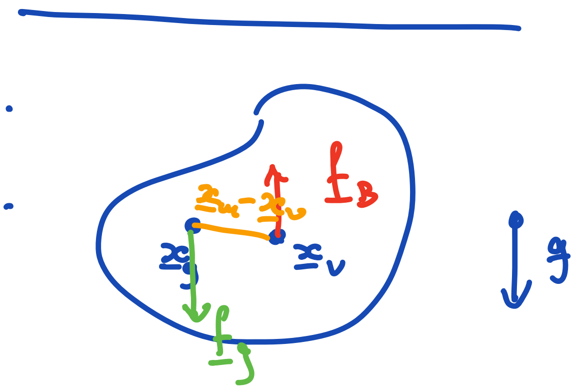
- Hydrostatic Eqbm:

$$\underline{f} = \underline{f}_G + \underline{f}_B = (m_b = m_f)\, g = 0$$

Today: - Isostacy

- Index notation

Hyrostatic force balance



$\underline{\tau}_G = \underline{x}_m \times m_B \underline{g}$ mom. grav.

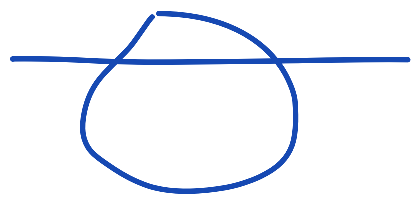
$\underline{\tau}_B = -\underline{x}_v \times m_f \underline{g}$ mom. buoy.

Hydrost: eqbm $\quad m_B = m_f \equiv m$

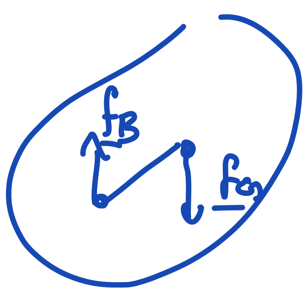
Total torque:

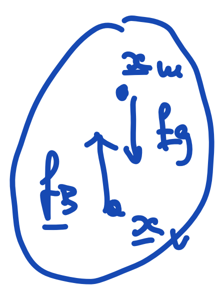
$$\underline{\tau} = \underline{\tau}_G + \underline{\tau}_B = \underline{x}_m \times m\underline{g} - \underline{x}_v \times m\underline{g}$$

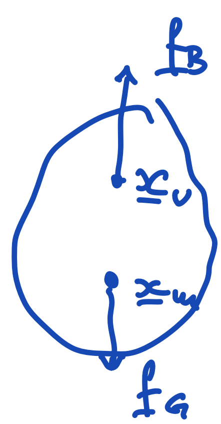
$$\boxed{\underline{\tau} = (\underline{x}_m - \underline{x}_v) \times m\,\underline{g}}$$



Stability of submerged body







$\underline{\tau} \neq 0$ unstable

$\underline{\tau} = 0$ meta stable

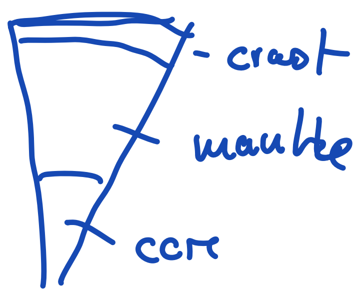
$\underline{\tau} = 0$ stable

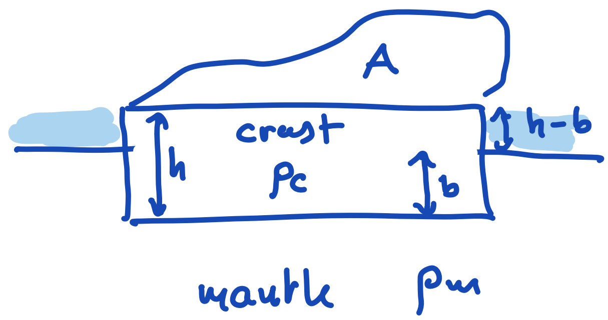
$\Rightarrow$ hydrostatic stability

Isostacy

Application of hydrostatic eqbm to Earth.

$$\rho_c < \rho_m < \rho_{core}$$





Force balance: $f = f_G + f_B = (m_c - m_m)\,g = 0$

mass of crust: $m_c = \rho_c h A$

mass of disp. mantle: $m_m = \rho_m b A$

$m_c = m_m \qquad \rho_c h \cancel{A} = \rho_m b \cancel{A}$

$$b = \frac{\rho_c}{\rho_m} h$$

depth of ocean: $h - b = \left(1 - \frac{\rho_c}{\rho_m}\right) h$

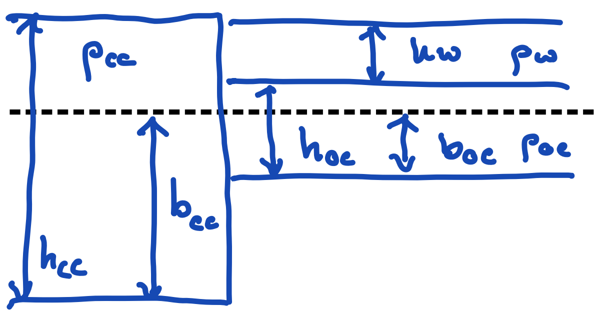
Example: $h \sim 35$ km

$$\left.\begin{array}{l} \rho_c \sim 2750 \ \frac{kg}{m^3} \\ \rho_m \sim \underset{\substack{\uparrow \\ 3 \\ \text{mean}}}{3}500 \ \frac{kg}{m^3} \end{array}\right\} \ \frac{\rho_c}{\rho_m} \sim 0.83$$

$\Rightarrow h - b = 5.8$ km

Observed depth of ocean ~ 4 km

More realistic model includes oceanic crust & water.



$\rho_{oc} \sim 2900 \ \frac{kg}{m^3}$

$h_{oc} \sim 6 \ km$

$\rho_w \sim 1000 \ \frac{kg}{m^3}$

$A_c \sim$ area of continent

$A_o \sim$ area of ocean

Geometric constraint:

$$h_{cc} - b_{cc} = h_{oc} + h_w - b_{oc}$$

Force balance on continent:

$$m_{cc} = m_m \qquad \rho_{cc} h_{cc} A_c = \rho_m b_{cc} A_c$$

$$\Rightarrow b_{cc} = \frac{\rho_{cc}}{\rho_m} h_{cc}$$

Force balance on ocean:

$$f = (m_{oc} + m_w - m_m) g = 0$$

$$\rho_{oc} h_{oc} \cancel{A_o} + \rho_w h_w \cancel{A_o} = \rho_m b_{oc} \cancel{A_o}$$

3 equations for 3 unknowns: h_w

1) $\rho_{cc} h_{cc} = \rho_m b_{cc}$

2) $\rho_{oc} h_{oc} + \rho_w h_w = \rho_m b_{oc}$

3) $h_{cc} - b_{cc} = h_w + h_{oc} - b_{oc}$

solve for $\boxed{h_w = \frac{\rho_{cc} - \rho_m}{\rho_w - \rho_m} h_{cc} - \frac{\rho_{oc} - \rho_m}{\rho_w - \rho_m} h_{oc}}$ ~ 6.6 km

Finish index notation from lecture 1

Dummy index - summation convention

$$\underline{a} \cdot \underline{b} = \sum_{i=1}^{3} a_i b_i = a_i b_i$$

Free index $\quad a_j = c_i b_i c_j$

$$\{\underline{e}_i\} = \{\underline{e}_1 \ \underline{e}_2 \ \underline{e}_3\}$$

Kronecker delta: $\{\underline{e}_i\}$

$$\delta_{ij} = \underline{e}_i \cdot \underline{e}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\delta_{ij} = \delta_{ji}$$

$$\underline{e}_i = \delta_{ij}\,\underline{e}_j$$

Scalar product: $\underline{a} = a_i\,\underline{e}_i$

$\underline{b} = b_j\,\underline{e}_j$

$$\underline{a}\cdot\underline{b} = (a_i\,\underline{e}_i)\cdot(b_j\,\underline{e}_j) = a_i\,b_j\,\underbrace{\underline{e}_i\cdot\underline{e}_j}_{\delta_{ij}}$$

$$= \delta_{ij}\,a_i\,b_j$$

$$= a_i\,b_i = a_j\,b_j$$

Permutation symbol (Levi-Civita)

$$\boxed{\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \in \{123,\ 231,\ 312\} \text{ even perm. of } 123 \\ -1 & \text{if } ijk \in \{321,\ 213,\ 132\} \text{ odd perm.} \\ 0 & \text{if any repeated index} \end{cases}}$$

123   213   231   113

Flipping indices changes sign

$$\epsilon_{ijk} = -\epsilon_{kji} = -\epsilon_{ikj} = -\epsilon_{jik}$$

Invariant under cyclic permutation

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$$

Alternative definitions

$$\boxed{\begin{aligned} \epsilon_{ijk} &= (\underline{e}_i \times \underline{e}_j) \cdot \underline{e}_k \\ \epsilon_{ijk} &= \det([\underline{e}_i \; \underline{e}_j \; \underline{e}_k]) \end{aligned}}$$

For orthonormal frame

$$\underline{e}_i \times \underline{e}_j = \epsilon_{ijk} \, \underline{e}_k$$

$$\underline{e}_1 \times \underline{e}_2 = \underbrace{\epsilon_{123}}_{1} \, \underline{e}_3$$

$$\underline{e}_2 \times \underline{e}_1 = \underbrace{\epsilon_{213}}_{-1} \, \underline{e}_3$$

Vector product: $\underline{a} \times \underline{b} = \underline{c}$

$\underline{a} = a_i \underline{e}_i \qquad \underline{b} = b_j \underline{e}_j \qquad \underline{c} = c_k \underline{e}_k$

$$\underline{a} \times \underline{b} = (a_i \underline{e}_i) \times (b_j \underline{e}_j) = a_i b_j \underbrace{\underline{e}_i \times \underline{e}_j}_{\epsilon_{ijk} \underline{e}_k}$$

$$= \epsilon_{ijk} a_i b_j \, \underline{e}_k$$

$$\Rightarrow \boxed{c_k = \epsilon_{ijk} a_i b_j}$$

Triple scalar product: $\underbrace{(\underline{a} \times \underline{b})}_{} \cdot \underline{c} = \boxed{\alpha}$

$\underline{a} = a_i \underline{e}_i \quad \underline{b} = b_j \underline{e}_j \quad \underline{d} = d_k \underline{e}_k \quad \underline{c} = c_m \underline{e}_m$

$\underline{d} = \epsilon_{ijk} a_i b_j e_k$

$$\alpha = \underline{d} \cdot \underline{c} = (\epsilon_{ijk} a_i b_j \underline{e}_k) \cdot (c_m \underline{e}_m)$$

$$= \epsilon_{ijk} a_i b_j c_m \underbrace{\underline{e}_k \cdot \underline{e}_m}_{\delta_{km}}$$

$$= \epsilon_{ijk} a_i b_j c_m \delta_{km}$$

$$\boxed{\alpha = \underbrace{\epsilon_{ijk}}_{1} a_i b_j c_k}$$

$i=1 \quad j=2 \quad k=3$

Frame identities

$$\boxed{\underline{e}_j = \delta_{ij} \underline{e}_i \qquad \underline{e}_i \times \underline{e}_j = \epsilon_{ijk} \underline{e}_k}$$

consequence of orthonormal frame

Epsilon-delta identities

in any right handed frame

$$\boxed{\begin{aligned} \epsilon_{pqs}\,\epsilon_{nrs} &= \delta_{pn}\,\delta_{qr} - \delta_{pr}\,\delta_{qn} \\ \epsilon_{pqs}\,\epsilon_{rqs} &= 2\,\delta_{pr} \end{aligned}}$$

helpful in vector (calc) identities

Example: $\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a}\cdot\underline{c})\,\underline{b} - (\underline{a}\cdot\underline{b})\,\underline{c} = \underline{d}$

$\underline{b} = b_i\,\underline{e}_i \qquad \underline{c} = c_j\,\underline{e}_j \qquad \underline{a} = a_q\,\underline{e}_q \qquad \underline{d} = d_p\,\underline{e}_p$

$\underline{b}\times\underline{c} = \epsilon_{ijk}\, b_i\, c_j\, \underline{e}_k$

$(a_q\,\underline{e}_q) \times (\epsilon_{ijk}\, b_i\, c_j\, \underline{e}_k) = \epsilon_{ijk}\, a_q\, b_i\, c_j\;\; \underline{e}_q \times \underline{e}_k$

$\underline{e}_q \times \underline{e}_k = \epsilon_{qkp}\;\underline{e}_p$

subst.

$$= \epsilon_{ijk}\, a_q\, b_i\, c_j\;\; \epsilon_{qkp}\,\underline{e}_p$$

$$= \underbrace{\epsilon_{ijk}\,\epsilon_{qkp}}\, a_q\, b_i\, c_j\;\; \underline{e}_p \qquad \epsilon_{qkp} = -\epsilon_{qpk} = \epsilon_{pqk}$$

$$= \epsilon_{ijk}\,\epsilon_{pqk}\, a_q\, b_i\, c_j\, \underline{e}_p$$

$$= (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})\, a_q\, b_i\, c_j\, \underline{e}_p$$

$$= \delta_{ip}\delta_{jq}\, a_q\, b_i\, c_j\, \underline{e}_p - \delta_{iq}\delta_{jp}\, a_q\, b_i\, c_j\, \underline{e}_p$$

$$= a_j\, b_i\, c_j\, \underline{e}_i \;-\; a_i\, b_i\, \underbrace{c_j\, \underline{e}_j}_{\underline{c}}$$

$$= a_j\, c_j\, (b_i\, \underline{e}_i)$$

$$(\underline{a}\cdot\underline{c})\,\underline{b} \;-\; (\underline{a}\cdot\underline{b})\,\underline{c}$$