Lecture 4: Introduction to Tensors

Logistics: Office hrs:

Mon 12-1 pm Afzal on zoom

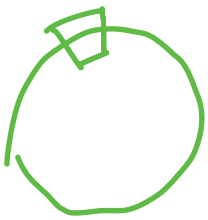
Tue 4-5 pm Marc on zoom

$\Rightarrow$ links will be posted on website

- post HW 1 today due next Th

Last time: - Hydrostatic eqbm

- Isostacy



$\Rightarrow$ depth of ocean basins

Force balance: $\underline{f} = -\underline{f}_G + \underline{f}_B = (m_c - m_m)\underline{g} = 0$

- Finished index notation

Permutation symbol: ϵ_{ijk}

$\Rightarrow$ cross product: $\underline{c} = \underline{a} \times \underline{b}$

$c_k = \epsilon_{ijk} a_i b_j$

$\epsilon\delta$-identities $\epsilon\epsilon \rightarrow \delta\delta - \delta\delta$

Today: Tensors

Introduction to Tensors

scalars: quantity $T(\underline{x})$, $p(\underline{x})$

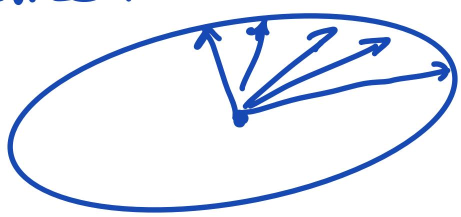
vectors: quantity + direction

velocity speed + direction

tensor: describes how a quantity changes with direction

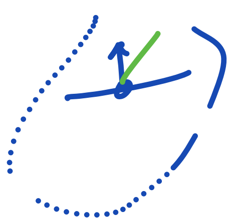
material properties → anisotropy

can be visualized as ellipsoid



examples: thermal conductivity

stress and strain



Second-order Tensors

Linear operator: $\boxed{\underline{v} = \underline{\underline{A}}\,\underline{u}}$

maps $\underline{u}$ into $\underline{v}$ $\quad \underline{u}, \underline{v} \in \mathcal{V}$

Two tensors are equal

$$\underline{\underline{A}}\,\underline{v} = \underline{\underline{B}}\,\underline{v} \quad \text{for all } \underline{v} \in \mathcal{V}$$

Zero tensor: $\underline{\underline{0}}\,\underline{v} = \underline{0}$ for all $\underline{v} \in \mathcal{V}$

Identity tensor: $\underline{\underline{I}}\,\underline{v} = \underline{v}$ for all $\underline{v} \in \mathcal{V}$

Tensor algebra

α = scalar $\quad \underline{v}$ = vector $\quad \underline{\underline{A}}, \underline{\underline{B}}$ = tensors

1) $(\alpha \underline{\underline{A}})\,\underline{v} = \underline{\underline{A}}\,(\alpha \underline{v})$ scalar multiplication
2) $(\underline{\underline{A}} + \underline{\underline{B}})\,\underline{v} = \underline{\underline{A}}\,\underline{v} + \underline{\underline{B}}\,\underline{v}$ tensor sum
3) $(\underline{\underline{A}}\,\underline{\underline{B}})\,\underline{v} = \underline{\underline{A}}\,(\underline{\underline{B}}\,\underline{v})$ tensor product
4) tensor scalar product $\rightarrow$ later

1+2 imply linearity

1, 2, 3 all give another tensor $\Rightarrow$ vector space

Representation of a tensor

frame $\{ \underline{e}_i \}$

$\underline{\underline{S}}$ is represented by 9 numbers/components

$$\boxed{S_{ij} = \underline{e}_i \cdot \underline{\underline{S}}\, \underline{e}_j}$$

$$S_{12} = \underline{e}_1 \cdot (\underline{\underline{S}}\, \underline{e}_2)$$

Matrix representation

$$[\underline{\underline{S}}] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

Consider: $\underline{v} = \underline{\underline{S}}\, \underline{u}$

$$\underline{v} = v_k \underline{e}_k \qquad \underline{u} = u_j \underline{e}_j$$

$$\underline{v} = \underline{\underline{S}}\, \underline{u}$$

$$v_k \underline{e}_k = \underline{\underline{S}} (u_j \underline{e}_j) = u_j\, \underline{\underline{S}}\, \underline{e}_j$$

multiply by $\underline{e}_i$ from left

$$v_k \underbrace{\underline{e}_i \cdot \underline{e}_k}_{\delta_{ik}} = u_j \underbrace{\underline{e}_i \cdot \underline{\underline{S}}\, \underline{e}_j}_{S_{ij}} = S_{ij} u_j$$

$$\boxed{v_i = S_{ij}\, u_j}$$

$$v_1 = \sum_{j=1}^{3} (S_{1j}\, u_j), \quad v_2 = \sum_{j=1}^{3} S_{2j}\, u_j, \quad v_3 = \sum_{j=1}^{3} S_{3j}\, u_j$$

Dyadic product

two vectors $\underline{a}$ & $\underline{b}$

$$\boxed{(\underline{a} \otimes \underline{b})\,\underline{v} = \underbrace{(\underline{b}\cdot\underline{v})\,\underline{a}}_{\text{vector}}} \quad \text{for all } \underline{v} \in \mathcal{V}$$

$$\underline{\underline{A}}\,\underline{v} = \alpha\, \underline{a}$$

$$[\underline{a} \otimes \underline{b}]_{ij}\, v_j = b_j\, v_j\, a_i$$

$$[\underline{a} \otimes \underline{b}]_{ij}\, v_j = (a_i\, b_j)\, v_j$$

by comparison: $[\underline{a} \otimes \underline{b}]_{ij} = a_i\, b_j$

$$\boxed{[\underline{a} \otimes \underline{b}] = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}}$$

Note: some books $\underline{a} \otimes \underline{b} = \underline{a}\underline{b}$

have to be careful if all vectors are column vec

$$\underline{a} \cdot \underline{b} = \underset{1 \cdot 3}{\underline{a}^T} \; \underset{3 \cdot 1}{\underline{b}}$$

Notions

$$\underset{3 \cdot 3}{\underline{a} \otimes \underline{b}} = \underset{3 \cdot 1}{\underline{a}} \; \underset{1 \cdot 3}{\underline{b}^T} =$$

Linearity of dyadic product

$\alpha, \beta \in \mathbb{R}$ vectors $\underline{a}, \underline{b}, \underline{v}, \underline{w} \in \mathcal{V}$

$$\boxed{(\underline{a} \otimes \underline{b})(\alpha \underline{v} + \beta \underline{w}) = \alpha (\underline{a} \otimes \underline{b}) \underline{v} + \beta (\underline{a} \otimes \underline{b}) \underline{w}}$$

$$\underline{\underline{A}}(\alpha \underline{v} + \beta \underline{w}) = \alpha \underline{\underline{A}}\, \underline{v} + \beta \underline{\underline{A}}\, \underline{w}$$

Product of dyadic products

$$\boxed{(\underline{a} \otimes \underline{b})(\underline{c} \otimes \underline{d}) = (\underline{b} \cdot \underline{c})\, \underline{a} \otimes \underline{d}} \Rightarrow \text{HW}$$

Basis for $\mathcal{V}^2$

Given frame $\{\underline{e}_i\}$ the nine dyadic products $\{\underline{e}_i \otimes \underline{e}_j\}$ form basis for $\mathcal{V}^2$.

$$\boxed{\underline{\underline{S}} = S_{ij}\, \underline{e}_i \otimes \underline{e}_j} \qquad \text{where} \qquad S_{ij} = \underline{e}_i \cdot \underline{\underline{S}}\, \underline{e}_j$$

$$\underline{\underline{S}} = \sum_{i=1}^{3} \sum_{j=1}^{3} S_{ij}\, \underline{e}_i \otimes \underline{e}_j$$

$$\underline{e}_1 \otimes \underline{e}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad 13$$

$$\underline{e}_2 \otimes \underline{e}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{v} = \underline{\underline{S}}\,\underline{u} \qquad \underline{v} = v_i \underline{e}_i \qquad \underline{u} = u_k \underline{e}_k$$

$$v_i \underline{e}_i = S_{ij} (\underline{e}_i \otimes \underline{e}_j)(u_k \underline{e}_k)$$

$$= S_{ij} u_k \, (\underline{e}_i \otimes \underline{e}_j)\underline{e}_k \qquad \text{apply dyadic prop.}$$

$$= S_{ij} u_k \, \underbrace{(\underline{e}_j \cdot \underline{e}_k)}_{\delta_{jk}} \underline{e}_i$$

$$= S_{ij} u_k \, \delta_{jk}\, \underline{e}_i \qquad \text{transfer prop}$$

$$v_i \underline{e}_i = S_{ij} u_j \underline{e}_i \quad \Rightarrow \quad \boxed{v_i = S_{ij} u_j}$$

<u>Tensor algebra in components</u>

Tensor addition: $\underline{\underline{H}} = \underline{\underline{S}} + \underline{\underline{T}}$

$$H_{ij} \underline{e}_i \otimes \underline{e}_j = S_{ij} (\underline{e}_i \otimes \underline{e}_j) + T_{ij} (\underline{e}_i \otimes \underline{e}_j)$$

$$= (S_{ij} + T_{ij}) \underline{e}_i \otimes \underline{e}_j$$

$$\boxed{H_{ij} = S_{ij} + T_{ij}}$$

Tensor product: $\underline{\underline{H}} = \underline{\underline{S}}\,\underline{\underline{T}}$

$$\underline{\underline{H}} = S_{ij} (\underline{e}_i \otimes \underline{e}_j) T_{kl} (\underline{e}_k \otimes \underline{e}_l)$$

$$= S_{ij} T_{kl} \underbrace{(\underline{e}_i \otimes \underline{e}_j)(\underline{e}_k \otimes \underline{e}_l)}_{\underbrace{(\underline{e}_j \cdot \underline{e}_k)}_{\delta_{jk}} \underline{e}_i \otimes \underline{e}_l}$$

$$=$$

$$= S_{ij} T_{kl} \delta_{jk} \, \underline{e}_i \otimes \underline{e}_l$$

$$= S_{ij} T_{jl} \, \underline{e}_i \otimes \underline{e}_l$$

$$H_{il} (\underline{e}_i \otimes \underline{e}_l) = S_{ij} T_{jl} \, \underline{e}_i \otimes \underline{e}_l$$

$$\boxed{H_{il} = S_{ij} T_{jl}}$$

$$H_{23} = \sum_{j=1}^{3} S_{2j} T_{j3}$$

Transpose of Tensor

$$\boxed{\underline{\underline{S}}\,\underline{u} \cdot \underline{v} = \underline{u} \cdot \underline{\underline{S}}^T \underline{v}} \quad \text{for all } \underline{u}, \underline{v} \in \mathcal{V}$$

this implies $\quad S^T_{ij} = S_{ji}$

$$(S_{ij}\, u_j\, \underline{e}_i) \cdot (v_l\, \underline{e}_l) = (u_k\, \underline{e}_k) \cdot (S^T_{ij}\, v_j\, \underline{e}_i)$$

$$S_{ij}\, u_j\, v_l\, \underbrace{(\underline{e}_i \cdot \underline{e}_l)}_{\delta_{il}} = S^T_{ij}\, u_k\, v_j\, \underbrace{(\underline{e}_k \cdot \underline{e}_i)}_{\delta_{ki}}$$

$$S_{ij}\, u_j\, v_i = S^T_{ij}\, u_i\, v_j \qquad \text{rename } i \to j, \; j \to i$$

$$S_{ij}\, u_j\, v_i = S^T_{ji}\, u_j\, v_i$$

$$\Rightarrow \quad S_{ij} = S^T_{ji}$$

Properties of Transpose

$$\boxed{\begin{aligned} (\underline{\underline{A}}^T)^T &= \underline{\underline{A}} \\ (\underline{\underline{A}}\,\underline{\underline{B}})^T &= \underline{\underline{B}}^T \underline{\underline{A}}^T \\ (\underline{u} \otimes \underline{v})^T &= \underline{v} \otimes \underline{u} \end{aligned}}$$

$\underline{\underline{S}}$ is symmetric if $\underline{\underline{S}} = \underline{\underline{S}}^T$ $\quad S_{ij} = S_{ji}$

$\underline{\underline{S}}$ is skewsym. if $\underline{\underline{S}} = -\underline{\underline{S}}^T$ $\quad S_{ij} = -S_{ji}$

Symmetric-Skew decomposition

$$\boxed{\begin{aligned} \underline{\underline{S}} &= \underline{\underline{E}} + \underline{\underline{W}} \\ \underline{\underline{E}} &= \tfrac{1}{2}(\underline{\underline{S}} + \underline{\underline{S}}^T) \\ \underline{\underline{W}} &= \tfrac{1}{2}(\underline{\underline{S}} - \underline{\underline{S}}^T) \end{aligned}}$$

$$\underline{\underline{E}} = \underline{\underline{E}}^T$$

$$\underline{\underline{W}} = -\underline{\underline{W}}^T$$

Trace of Tensor

$$tr(\underline{a} \otimes \underline{b}) = \underline{a} \cdot \underline{b} = a_i b_i$$

this implies

$$\boxed{tr(\underline{\underline{A}}) = A_{ii} = A_{11} + A_{22} + A_{33}}$$

$$\mathrm{tr}(\underline{\underline{A}}) = \mathrm{tr}(A_{ij}\, \underline{e}_i \otimes \underline{e}_j) = A_{ij}\, \mathrm{tr}(\underline{e}_i \otimes \underline{e}_j)$$

$$= A_{ij}\, \underbrace{(\underline{e}_i \cdot \underline{e}_j)}_{\delta_{ij}}$$

$$= A_{ii}$$

Properties: $\mathrm{tr}(\underline{\underline{A}}^T) = \mathrm{tr}(\underline{\underline{A}})$

$$\mathrm{tr}(\underline{\underline{A}}\,\underline{\underline{B}}) = \mathrm{tr}(\underline{\underline{B}}\,\underline{\underline{A}})$$

$$\mathrm{tr}(\underline{\underline{A}} + \underline{\underline{B}}) = \mathrm{tr}(\underline{\underline{A}}) + \mathrm{tr}(\underline{\underline{B}})$$

$$\mathrm{tr}(\alpha\, \underline{\underline{A}}) = \alpha\, \mathrm{tr}(\underline{\underline{A}})$$

Still missing: 1) Tensor scalar product

2) determinant