Lecture 5: Cauchy Stress Tensor

Logistics: - HW1 due Thurs. at 9^{30} am
- Office hours 4-5 pm today on zoom

Last time: - Second-order Tensors $\quad \underline{v} = \underline{\underline{A}}\,\underline{u}$
- Dyadic product: $\boxed{(\underline{a} \otimes \underline{b})\underline{v} = (\underline{b} \cdot \underline{v})\,\underline{a}}$
- Tensor basis: $\underline{\underline{S}} = \sum_i \sum_j S_{ij}\, \underline{e}_i \otimes \underline{e}_j$

$$S_{ij} = \underline{e}_i \cdot \underline{\underline{S}}\, \underline{e}_j$$

$$\boxed{[\underline{a} \otimes \underline{b}] = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}}$$

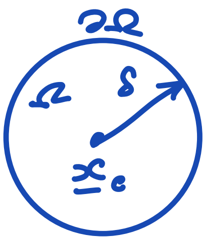
$\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{H}}$

- Tensor algebra $\alpha(\underline{\underline{A}} + \underline{\underline{B}})\underline{b} = \alpha \underline{\underline{A}}\,\underline{b} + \alpha \underline{\underline{B}}\,\underline{b}$
- Tensor properties: Transpose, trace

Today: - Cauchy stress tensor
- Cauchy's postulate
- Action - Reaction
- Cauchy's Theorem

Cauchy-Stress Tensor

Force balance on in limit of small body



$$\lim_{\delta \to 0} \underline{f} = m\,\underline{a}$$

$$m = \int_\Omega \rho \, dV$$

V_Ω = Volume

A_Ω = surf. area

$$\underline{f} = \underline{F}_b[\Omega] + \underline{F}_s[\partial\Omega] = \int_\Omega \rho \underline{g} \, dV + \oint_{\partial\Omega} \underline{t} \, dS$$

$$\lim_{\delta \to 0} m = \int_\Omega \rho(\underline{x}) \, dV \approx \rho(\underline{x}_0) \int_\Omega dV = \rho_0 V_\Omega$$

$$\lim_{\delta \to \delta} \underline{F}_b[\Omega] = \int_\Omega \rho \underline{g} \, dV \approx \rho_0 \underline{g}_0 V_\Omega$$

Substitute $\underline{F}_s[\partial\Omega] = m\,\underline{a} - \underline{F}_b[\Omega]$

$$\lim_{\delta \to 0} \oint_{\partial\Omega} \underline{t} \, dA = \int_\Omega \rho \underline{a} - \rho \underline{g} \, dV \approx \rho_0 (\underline{a}_0 - \underline{g}_0) V_\Omega$$

divide by area

$$\lim_{\delta \to 0} \frac{1}{A_\Omega} \oint_{\partial\Omega} \underline{t} \, dA = \frac{V_\Omega}{A_\Omega} \rho_0 (\underline{a}_0 - \underline{g}_0)$$

Volume vanishes faster than surface are

$$\lim_{\delta \to 0} \frac{V_\Omega}{A_\Omega} = 0$$

Consider sphere: $V_\Omega = \frac{4}{3}\pi\delta^3$ $A_\Omega = 4\pi\delta^2$

$$\lim_{\delta \to 0} \frac{V_\Omega}{A_\Omega} = \frac{\delta}{3} = 0$$

$\Rightarrow$ This also holds for all other bodies

Force balance on infinitesimal body

$$\boxed{\lim_{\delta \to 0} \frac{1}{A_\Omega} \oint_{\partial\Omega} \underline{t}\, dA = \underline{0}}$$

Note: • $\frac{1}{A_\Omega}$ normalization

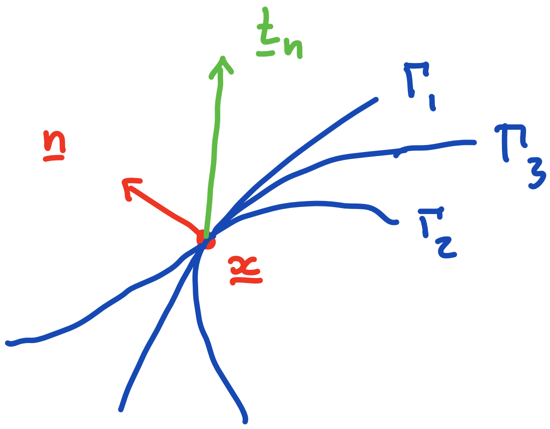
• assumed ρ, $|\underline{g}|$, $|\underline{a}|$, $|\underline{t}|$

are all finite and continuous

$\Rightarrow$ basis for derivation of Cauchy-stress tensor

Cauchy's postulate

The traction field, $\underline{t}_n$ on surface Γ depends only pointwise on the unit normal vector $\underline{n}$. In particular there is a traction field s.t. $\boxed{\underline{t}_n = \underline{t}_n(\underline{n}(\underline{x}), \underline{x})}$



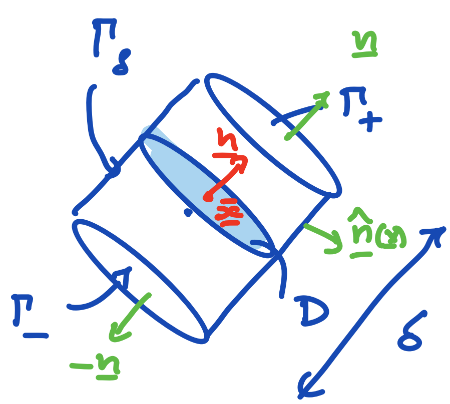
$\Rightarrow$ $\underline{t}_n$ is independent of $\nabla \underline{n}$ and hence the curvature of surface.

Law of Action - Reaction

If $\underline{t}_n$ is continuous & bounded

$$\boxed{\underline{t}_n(-\underline{n}, \underline{x}) = -\underline{t}_n(\underline{n}, \underline{x})}$$

Disk of area D



$$\partial\Omega = \Gamma^+ \cup \Gamma^- \cup \Gamma_\delta$$

resultant surface force:

$$\underline{f}_s[\partial\Omega] = \int_{\partial\Omega} \underline{t}_n(n, \underline{x})\, dA$$

$$= \int_{\Gamma_s} \cancel{\underline{t}_n(\underline{\hat{n}}(\underline{x}), \underline{x})}^{\,0}\, dA + \int_{\Gamma_+} \underline{t}_n(\underline{n}, \underline{x})\, dA + \int_{\Gamma_-} \underline{t}_n(-\underline{n}, \underline{x})\, dA$$

$$\lim_{\delta \to 0} \frac{1}{A_\Omega} \underline{f}_s[\partial\Omega] = \frac{1}{A} \int_D \underline{t}_n(\underline{n}, \underline{x}) + \underline{t}_n(-\underline{n}, \underline{x})\, dA = 0$$

localization: Because the location & radius of disk are arbitrary $\Rightarrow$ integrand must be zero everywhere.

$$\Rightarrow \quad \underline{t}_n(\underline{n}, \underline{x}) + \underline{t}_n(-\underline{n}, \underline{x}) = 0$$

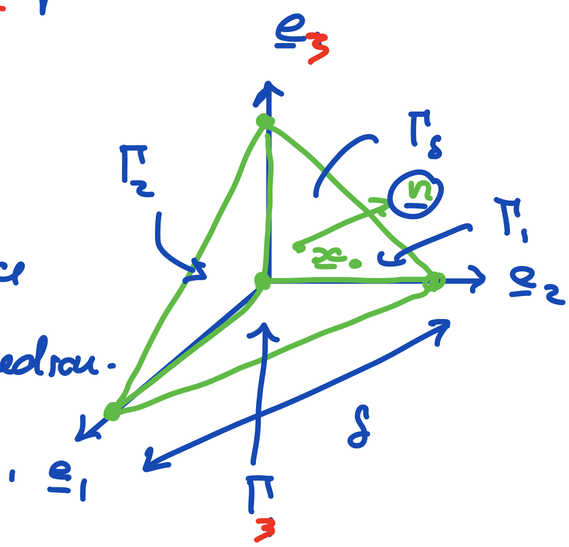
$$\underline{t}_n(-n, \underline{x}) = -\underline{t}_n(\underline{n}, \underline{x}) \quad \checkmark$$

Cauchy's Theorem

Let $\underline{t}(\underline{n}, \underline{x})$ satisfy Cauchy's postulate. Then $\underline{t}(\underline{n}, \underline{x})$ is linear in $\underline{n}$, that is, for each $\underline{x}$ there is a second-order tensor field $\underline{\underline{S}}(\underline{x}) \in \mathcal{V}^2$ such that

$$\boxed{\underline{t}(\underline{n}, \underline{x}) = \underline{\underline{S}}(\underline{x})\, \underline{n}}$$

called the Cauchy stress field.



$\{\underline{e}_i\}$

$\underline{x}_0$ and $\underline{n}$ define a surface that defines a irregular tetrahedron.

δ lenght of max side

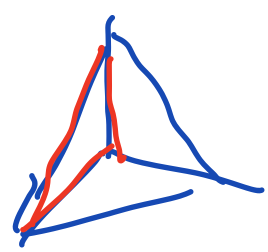
Surface area: $\partial\Omega = \Gamma_\delta \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$

normals on Γ_i is $\underline{n}_i = -\underline{e}_i$

outward

Force balance on Ω in limit of $\delta \to 0$

$$\lim_{\delta\to 0} \frac{1}{A_\Omega}\left[\oint_{\Gamma_\delta} \underline{t}(\underline{n},\underline{x})\, dA + \sum_{j=1}^{3} \oint_{\Gamma_j} \underline{t}(-\underline{e}_j,\underline{x})\, dA\right] = 0$$



Note: $n_j = \underline{n}\cdot\underline{e}_j \geq 0$

$A_{\Gamma_j} = n_j \cdot A_{\Gamma_\delta}$

$\Rightarrow$ future HW's

$$A_{\partial\Omega} = A_{\Gamma_\delta} + \sum_{j=1}^{3} \Gamma_j = \lambda A_{\Gamma_\delta} \qquad \lambda = 1 + \sum_{j=1}^{3} n_j$$

substitute

$$\lim_{\delta \to 0} \frac{1}{A_{\partial\Omega}} \left[\int_{\Gamma_\delta} \underline{t}(\underline{n}, \underline{x})\, dA + \sum_{j=1}^{3} \int_{\Gamma_\delta} \underline{t}_{\underline{n}}(-\underline{e}_j, \underline{x})\, n_j\, dA \right] = 0$$

$$\lim_{\delta \to 0} \frac{1}{A_{\partial\Omega}} \int_{\Gamma_\delta} \underline{t}(\underline{n}, \underline{x}) + \sum_{j=1}^{3} \underline{t}(-\underline{e}_j, \underline{x})\, n_j\, dA = 0$$

as the tetrahedron is arbitrary the integrand must be zero

$$\underline{t}(\underline{n}, \underline{x}) + \sum_{j=1}^{3} \underline{t}(-\underline{e}_j, \underline{x})\, n_j = \underline{0}$$

use law of action-reaction

$$\boxed{\underline{t}(\underline{n}, \underline{x}) = \sum_{j=1}^{3} \underline{t}(\underline{e}_j, \underline{x})\, n_j}$$

$$\underline{n} = n_i\, \underline{e}_i$$

Use definition of dyadic product

$$\begin{aligned}
(\underline{t}(\underline{e}_j, \underline{x}) \otimes \underline{e}_j)\, \underline{n} &= (\underline{e}_j \cdot \underline{n})\; \underline{t}(\underline{e}_j, \underline{x}) \\
&= \underline{e}_j \cdot (n_i\, \underline{e}_i)\; \underline{t}(\underline{e}_j, \underline{x}) \\
&= n_i\, \underbrace{\underline{e}_j \cdot \underline{e}_i}_{\delta_{ij}} \qquad ''
\end{aligned}$$

$$= n_j \, \underline{t}(\underline{e}_j, \underline{x})$$

So we have

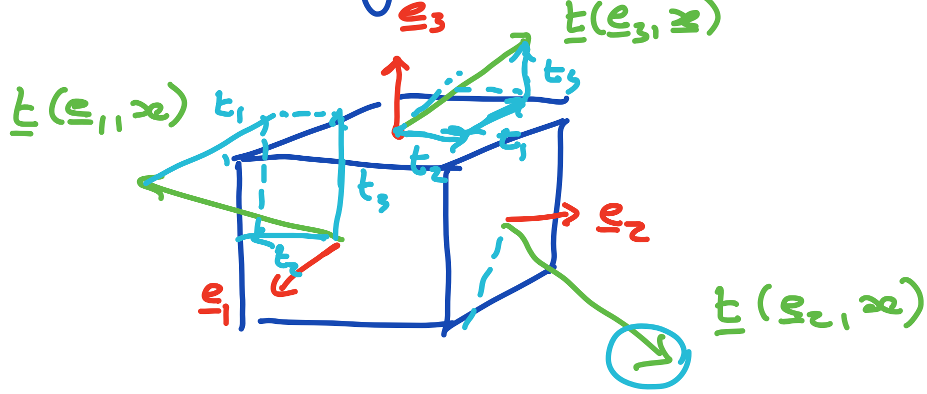
$$\underline{t}(\underline{n}, \underline{x}) = \underbrace{\left(\underline{t}(\underline{e}_j, \underline{x}) \otimes \underline{e}_j \right)}\, \underline{n}$$
$$= \underline{\underline{\sigma}}\, \underline{n}$$

$$\Rightarrow \qquad \underline{\underline{\sigma}} = \underline{t}(\underline{e}_j, \underline{x}) \otimes \underline{e}_j$$

traction: $\underline{t}(\underline{e}_j, \underline{x}) = t_i(\underline{e}_j, \underline{x})\, \underline{e}_i$ $\qquad \underline{a} = a_i \underline{e}_i$

$$\boxed{\begin{aligned} \underline{\underline{\sigma}} &= t_i(\underline{e}_j, \underline{x})\, \underline{e}_i \otimes \underline{e}_j \\ \sigma_{ij} &= t_i(\underline{e}_j, \underline{x}) \end{aligned}}$$

Hence σ_{ij} is the i-th component of the traction on the j-th coordinate plane.



$$\underline{\underline{\sigma}} = \left[\underline{t}(\underline{e}_1, \underline{x}) \quad \underline{t}(\underline{e}_2, \underline{x}) \quad \underline{t}(\underline{e}_3, \underline{x}) \right]$$

$$= \begin{bmatrix} t_1(\underline{e}_1, \underline{x}) & t_1(\underline{e}_2, \underline{x}) & t_1(\underline{e}_3, x) \\ t_2(\underline{e}_1, \underline{x}) & t_2(\underline{e}_2, x) & t_2(\underline{e}_3, x) \\ t_3(\underline{e}_1, \underline{x}) & t_3(\underline{e}_2, \underline{x}) & t_3(\underline{e}_3, \underline{x}) \end{bmatrix}$$

$$\underline{t}_n = \underline{\underline{\sigma}}\, \underline{n}$$