Lecture 6: More on stress

check points on canvas

Logistics: - HW 1 (5/7) please submit

- HW2 will be posted

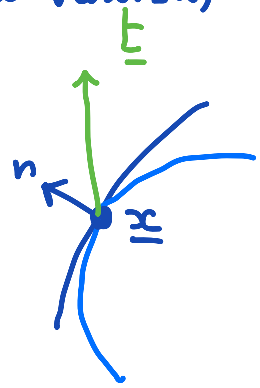
⇒ make use of office hours if HW is not clear

Last time: - infinitesimal force balance $f = \cancel{ma}$ $\quad f = F_s - \cancel{F_b}$

$$\frac{1}{A}\oint_{\partial\Omega} \underline{t}_n \, dA = \underline{0} \qquad \text{(Vol. terms vanish)}$$

- Cauchy's postulate

$$\boxed{\underline{t}_n = \underline{t}(\underline{n}, \underline{x})}$$



- 3rd law (Action-Reaction)

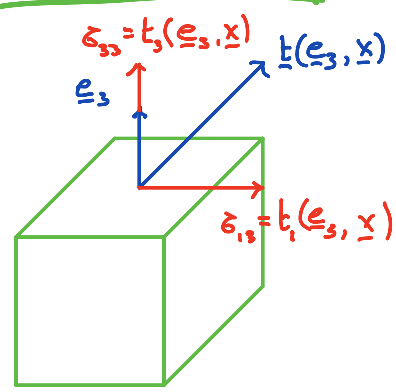
$$\boxed{\underline{t}_n(-\underline{n}, \underline{x}) = -\underline{t}(\underline{n}, \underline{x})}$$

- Cauchy's theorem $\boxed{\underline{t}_n = \underline{\underline{\sigma}}(\underline{x})\, \underline{n}}$

$\underline{\underline{\sigma}}$ = Cauchy stress

$$\boxed{\underline{\underline{\sigma}} = \sigma_{ij}\, \underline{e}_i \otimes \underline{e}_j}$$

$$\sigma_{ij} = t_i(\underline{e}_j, \underline{x})$$



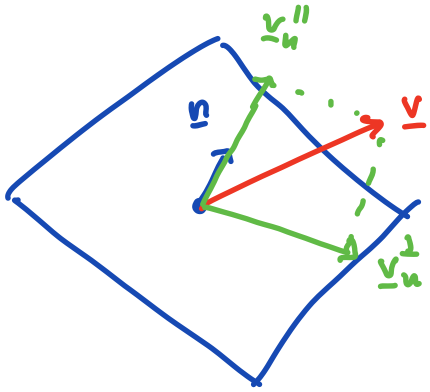
σ_{ij} = i-th component of traction on jth coordinate plane

Today: Projections → shear & normal stress

Simple stress states

Spherical + deviatoric stress

Projection Tensors



$$\underline{v} = \underline{v}_n^{\parallel} + \underline{v}_n^{\perp}$$

dot product

$$\underline{v}_n^{\parallel} = (\underline{n} \cdot \underline{v})\, \underline{n}$$

$$\underline{v}_n^{\perp} = \underline{v} - \underline{v}_n^{\parallel}$$

$$\underline{v}_n^{\parallel} = \underline{\underline{P}}_n^{\parallel}\ \underline{v}$$

$$\underline{v}_n^{\perp} = \underline{\underline{P}}_n^{\perp}\ \underline{v}$$

$\underline{\underline{P}}_n^{\parallel}$ & $\underline{\underline{P}}_n^{\perp}$ projection tensors

Use dyadic product: $(\underline{a} \otimes \underline{b})\, \underline{c} = (\underline{b} \cdot \underline{c})\, \underline{a}$

$(n \otimes n)\, v \quad n \quad v \quad n$

$$\underline{v}^{\parallel} = (\underline{n} \otimes \underline{n})\, \underline{v} = \underline{\underline{P}}_n^{\parallel}\, \underline{v}$$

$$\underline{v}^{\perp} = \underline{v} - \underline{v}_n^{\parallel} = \underline{\underline{I}}\, \underline{v} - (\underline{n} \otimes \underline{n})\, \underline{v} = (\underline{\underline{I}} - \underline{n} \otimes \underline{n})\, \underline{v}$$

$$= \underline{\underline{P}}_n^{\perp}\, \underline{v}$$

Projection tensors

$$\boxed{\underline{\underline{P}}_n^{\parallel} = \underline{n} \otimes \underline{n}} \qquad \boxed{\underline{\underline{P}}^{\perp} = \underline{\underline{I}} - \underline{n} \otimes \underline{n}}$$

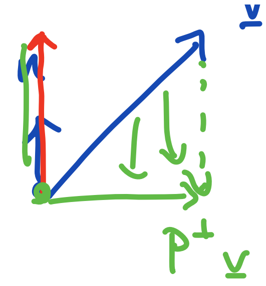
Properties:

$$\underline{\underline{P}} = \underline{\underline{P}}^T$$

$$\underline{\underline{P}}^2 = \underline{\underline{P}}$$

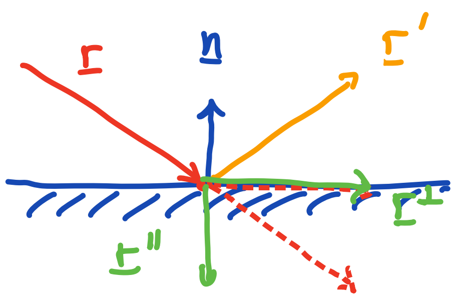
$$\underline{\underline{P}}^{\parallel} + \underline{\underline{P}}^{\perp} = \underline{\underline{I}}$$

$$\underline{\underline{P}}^{\parallel}\,\underline{\underline{P}}^{\perp} = \underline{\underline{0}}$$



$$\underline{\underline{P}}^{\perp}\underline{v}^{\parallel} = \underline{\underline{P}}^{\parallel}\underline{v}$$

Reflections



$$\underline{r}' = -\underline{r}^{\parallel} + \underline{r}^{\perp}$$

$$= \underline{r}^{\perp} - \underline{r}^{\parallel}$$

$$= \underline{\underline{P}}^{\perp}\underline{r} - \underline{\underline{P}}^{\parallel}\underline{r}$$

$$= (\underline{\underline{P}}^{\perp} - \underline{\underline{P}}^{\parallel})\,\underline{r}$$

$$= (\underline{\underline{I}} - \underline{n}\otimes\underline{n} - \underline{n}\otimes\underline{n})\,\underline{r}$$

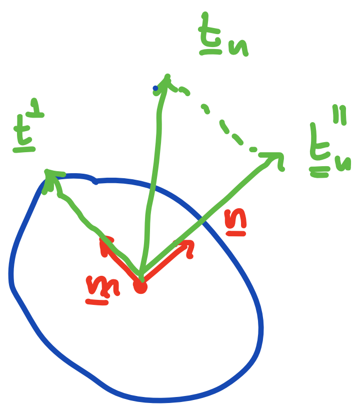
$$= (\underline{\underline{I}} - 2\,\underline{n}\otimes\underline{n})\,\underline{r}$$

$$= \underline{\underline{R}}$$

Fix notes

Reflection tensor: $\underline{\underline{R}} = \underline{\underline{I}} - 2\,\underline{n}\otimes\underline{n}$

Normal and Shear Stresses



$$\underline{t}_n = \underline{\underline{\sigma}}\,\underline{n}$$

$$\underline{\underline{P}}^{\parallel} = \underline{n} \otimes \underline{n}$$

$$\underline{\underline{P}}^{\perp} = \underline{\underline{I}} - \underline{n} \otimes \underline{n} = \underline{m} \otimes \underline{m}$$

normal stress: $\underline{t}_n^{\parallel} = \underline{\underline{P}}^{\parallel}\,\underline{t}_n = (\underline{n} \otimes \underline{n})\,\underline{t}_n$

$$= (\underline{n} \cdot \underline{t}_n)\,\underline{n} = \sigma_n\,\underline{n}$$

σ_n is mag. of normal stress

$$\sigma_n = \underline{n} \cdot \underline{t}_n = \underline{n} \cdot \underline{\underline{\sigma}}\,\underline{n} \qquad \sigma_n = n_i\,\sigma_{ij}\,n_j$$

shear stress: $\underline{t}_n^{\perp} = \underline{\underline{P}}^{\perp}\,\underline{t}_n = (\underline{m} \otimes \underline{m})\,\underline{t}_n = \underbrace{(\underline{m} \cdot \underline{t}_n)}_{\tau}\,\underline{m}$

mag. of shear stress: $\tau = \underline{m} \cdot \underline{t}_n = \underline{m} \cdot \underline{\underline{\sigma}}\,\underline{n} = m_i\,\sigma_{ij}\,n_j$

$\sigma_n > 0$ tensile stress

$\sigma_n < 0$ compressive stress

From geometry: $|\underline{t}_n|^2 = \sigma_n^2 + \tau^2$

Simple states of stress

I) Hydrostatic stress

$$\boxed{\underline{\underline{\sigma}} = -p\,\underline{\underline{I}}} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} \qquad \underline{t} = -p\,\underline{n}$$

$$\underline{t}_n = \underline{\underline{\sigma}}\,\underline{n} = (-p\,\underline{\underline{I}})\,\underline{n} = -p\,\underline{n}$$

$$\underline{t}^{\parallel} = \underline{\underline{P}}^{\parallel}\,\underline{t}_n = (\underline{n}\otimes\underline{n})(-p\,\underline{n}) = -p\,(\underline{n}\otimes\underline{n})\,\underline{n}$$

$$= \cancelto{1}{(\underline{n}\cdot\underline{n})}\,\underline{n}\,(-p) = -p\,\underline{n}$$

$$\underline{t}^{\parallel} = \underline{t}_n \quad \Rightarrow \quad \underline{t}^{\perp} = \underline{0}$$

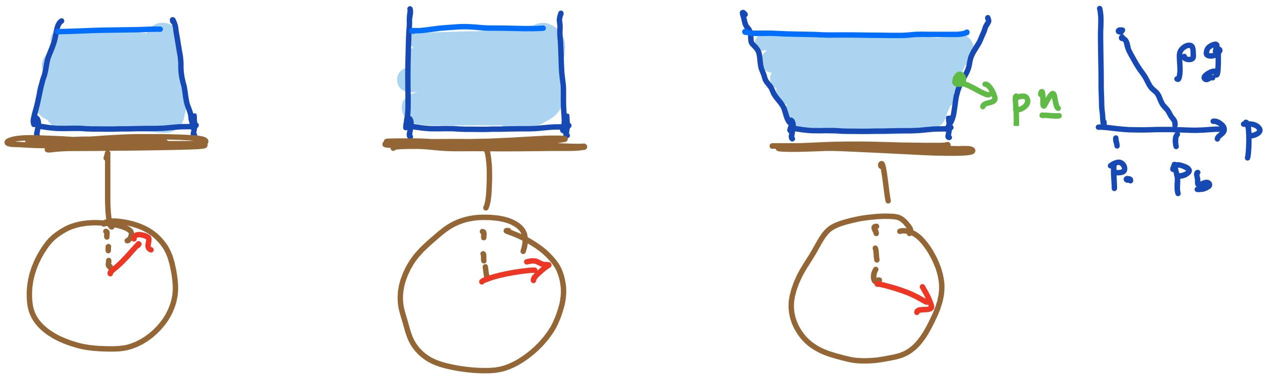
$$\left.\begin{array}{l} \text{normal stress: } \sigma_n = -p \\ \text{shear stress: } \tau = 0 \end{array}\right\} \text{ on all planes}$$

Pascal's law:

The pressure in a fluid at rest is independent of the direction of a surface.

Hydrostatic paradox:

force on base is same $\qquad F = A\,p$

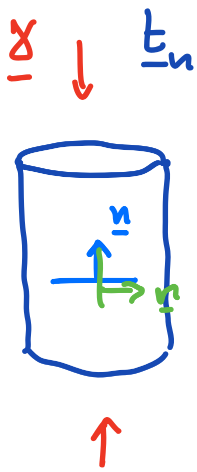


II Uniaxial stress

$$\boxed{\underline{\underline{\sigma}} = \sigma\ \underline{\gamma} \otimes \underline{\gamma}}$$

$\underline{\gamma}$ = unit vector

$$\underline{t}_n = \underline{\underline{\sigma}}\,\underline{n} = \sigma\,(\underline{\gamma}\otimes\underline{\gamma})\,\underline{n} = \sigma\,(\underline{\gamma}\cdot\underline{n})\,\underline{\gamma}$$



$$\underline{t}_n = \sigma\,(\underline{\gamma}\cdot\underline{n})\,\underline{\gamma}$$

$$\underline{t}_n = \pm\,\sigma\,\underline{\gamma}$$

$$\underline{t}_n = \sigma\,\cancel{(\underline{\gamma}\cdot\underline{n})}^{0}\,\underline{n} = \underline{0}$$

traction is always parallel to γ

III Pure shear stress

two perpendicular directions $\underline{\xi}$ and $\underline{\eta}$ $\quad \underline{\xi}\cdot\underline{\eta} = 0$

$$\boxed{\underline{\underline{\sigma}} = \tau\,(\underline{\xi}\otimes\underline{\eta} + \underline{\eta}\otimes\underline{\xi})}$$

$$\underline{t}_n = \underline{\underline{\sigma}}\,\underline{n} = \tau\left[(\underline{\xi}\otimes\underline{\eta})\underline{n} + (\underline{\eta}\otimes\underline{\xi})\underline{n}\right]$$

$$= \tau\left[(\underline{\eta}\cdot\underline{n})\underline{\xi} + (\underline{\xi}\cdot\underline{n})\underline{\eta}\right]$$

$\underline{n} = \underline{\eta}$: $\quad \underline{t}_n = \tau\left[\cancelto{1}{(\underline{\eta}\cdot\underline{\eta})}\,\underline{\xi} + \cancelto{0}{(\underline{\xi}\cdot\underline{\eta})}\,\underline{\eta}\right]$

$$= \tau\,\underline{\xi}$$

$\underline{n} = \underline{\xi}$: $\quad \underline{t}_n = \tau\,\underline{\eta}$

IV Plane stress

If there exist a pair of orthog. vectors $\underline{\xi}$ & $\underline{\eta}$ s.t. matrix representation in frame $\{\underline{\xi}, \underline{\eta}, \underline{\xi}\times\underline{\eta}\}$ is of form

$$[\underline{\underline{\sigma}}] = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Q: is uniaxial stress a plane stress ?

$$\boxed{\underline{\underline{\sigma}} = \sigma\, \underline{\gamma} \otimes \underline{\gamma}}$$

pick a frame $\{\underline{\gamma}, \underline{e}_2, \underline{e}_3\}$

$\underline{\gamma} \cdot \underline{e}_2 = 0 \quad \underline{\gamma} \cdot \underline{e}_3 = 0$

don't need to specify other basis vectors (not unique)

$$\underline{\underline{\sigma}} = \sigma_{ij}\, \underline{e}_i \otimes \underline{e}_j$$

$$\boxed{\sigma_{ij} = \underline{e}_i \cdot \underline{\underline{\sigma}}\, \underline{e}_j}$$

$$= \underline{e}_i \cdot (\sigma\, \underline{\gamma} \otimes \underline{\gamma}) \underline{e}_j \qquad \underline{\gamma} = \underline{e}_1$$

$$= \sigma\, \underline{e}_i \cdot (\underline{e}_1 \otimes \underline{e}_1) \underline{e}_j = \sigma\, \underline{e}_i \cdot (\underline{e}_1 \cdot \underline{e}_j)\, \underline{e}_1$$

$$\sigma_{ij} = \sigma\, (\underline{e}_1 \cdot \underline{e}_j)(\underline{e}_i \cdot \underline{e}_1)$$

$$\sigma_{11} = \sigma\, (\underline{e}_1 \cdot \underline{e}_1)(\underline{e}_1 \cdot \underline{e}_1) = \sigma$$

$$\sigma_{12} = \sigma\, \overset{0}{\cancel{(\underline{e}_1 \cdot \underline{e}_2)}} (\underline{e}_1 \cdot \underline{e}_1) = 0$$

$$\sigma_{21} = 0$$

$$[\underline{\underline{\sigma}}] = \begin{bmatrix} \boxed{\begin{matrix} \sigma & 0 \\ 0 & 0 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{bmatrix} \quad \checkmark \qquad \text{plane stress}$$

Spherical & Deviatoric Stress tensors

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}_S + \underline{\underline{\sigma}}_D \qquad\qquad \underline{\underline{\sigma}}'$$

spherical stress tensor: $\underline{\underline{\sigma}}_s = -p\,\underline{\underline{I}} \qquad p = -\frac{1}{3}\,\mathrm{tr}(\underline{\underline{\sigma}})$

deviatoric stress tensor: $\underline{\underline{\sigma}}_D = \underline{\underline{\sigma}} - \underline{\underline{\sigma}}_s$

$$= \underline{\underline{\sigma}} + p\,\underline{\underline{I}}$$

The pressure $p = -\frac{1}{3}\,\mathrm{tr}(\underline{\underline{\sigma}})$ is mean normal traction.

Sperical stress $\rightarrow$ volumetric changes

Deviatoric stress $\rightarrow$ changes shape of body