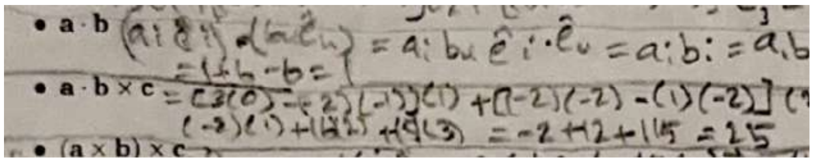
Lecture 7: Rotation, Change of basis, Eigenproblem

Logistics: - HW2 is due Th

- HW comments:
  - Start early / before office hrs
  - Please use <u>separate</u> sheet of paper!



  - In this class always start with base vectors a_i a_i

$\vec{a}\cdot(\underbrace{\vec{b}\times\vec{c}}_{d}) = a_i(\vec{b}\times\vec{c})_i$ correct but <u>shortcut</u>

$$(a_i \underline{e}_i)\cdot((b_m \underline{e}_m)\times(c_n \underline{e}_n))$$

$$\cdot\,(b_m c_n \; \underline{e}_m \times \underline{e}_n)$$

$$\cdot\,(\epsilon_{mnk} b_m c_n \, \underline{e}_k)$$

$$\epsilon_{mnk}\, a_i b_m c_n \underbrace{\underline{e}_i\cdot\underline{e}_k}_{\delta_{ik}}$$

$$= \epsilon_{mni}\, a_i b_m c_n$$

Last time: - Shear & normal stress

- Simple states of stress

Today: Rotations ....

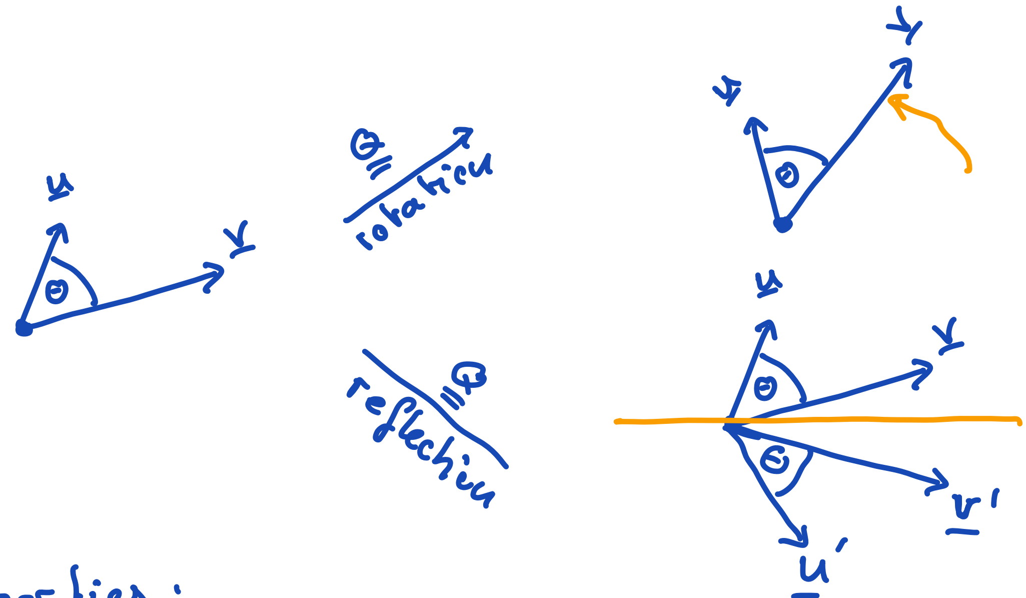
Orthogonal Tensors

defined by following transformation

$$\boxed{\underbrace{\underline{\underline{Q}}\,\underline{u}}_{\underline{u}'} \cdot \underbrace{\underline{\underline{Q}}\,\underline{v}}_{\underline{v}'} = \underline{u}\cdot\underline{v}} \qquad \text{for all } \underline{u}, \underline{v} \in \mathcal{V}$$

$$\underline{u}\cdot\underline{v} = |\underline{u}|\,|\underline{v}|\cos\theta$$

$\Rightarrow$ preserves the length of $\underline{u}$ & $\underline{v}$ and angle θ



Properties:

$$\boxed{\begin{aligned} \underline{\underline{Q}}^T &= \underline{\underline{Q}}^{-1} \\ \underline{\underline{Q}}^T\underline{\underline{Q}} &= \underline{\underline{I}} \\ \det(\underline{\underline{Q}}) &= \pm 1 \end{aligned}}$$

$\det(\underline{\underline{Q}}) = 1 \Rightarrow$ rotation

$\det(\underline{\underline{Q}}) = -1 \Rightarrow$ reflection

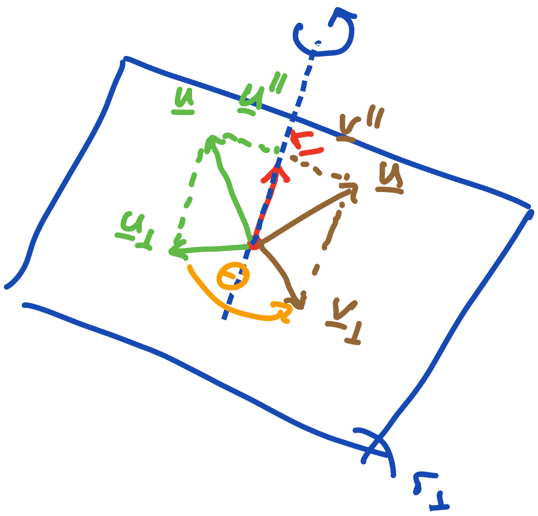
in mechanics mostly interested in rotations

Rotation Matrices

$\underline{v} = \underline{\underline{Q}}(\underline{r}, \theta)\ \underline{u}$

$\underline{r}$ = axis of rotation (unit vector)

θ = counterclockwise angle



$\underline{u} = \underline{u}^{\parallel} + \underline{u}^{\perp}$

$\underline{v} = \underline{v}^{\parallel} + \underline{v}^{\perp}$

$\underline{u}^{\parallel} = \underline{v}^{\parallel} = (\underline{u} \cdot \underline{r})\underline{r} = \underbrace{(\underline{r} \otimes \underline{r})}_{\underline{\underline{P}}^{\parallel}}\ \underline{u}$

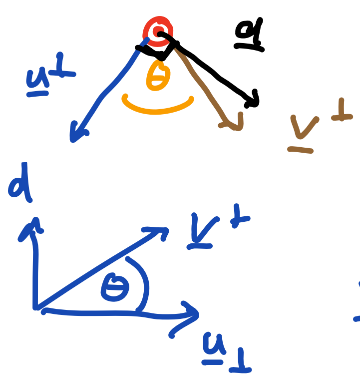
What is $\underline{v}^{\perp}$?

looking onto $r_{\perp}$

$\underline{u}^{\perp} = \underline{u} - \underline{u}^{\parallel} = (\underline{\underline{I}} - \underline{r} \otimes \underline{r})\,\underline{u}$

$\underline{d} = \underline{r} \times \underline{u}$

$\underline{v}_{\perp} = \cos\theta\ \underline{u}_{\perp} + \sin\theta\ \underline{d}$



Rotated vector

$\underline{v} = \underline{v}_{\parallel} + \underline{v}_{\perp} =$

$= (\underline{r} \otimes \underline{r})\underline{u} + \cos\theta\ \underline{u}_{\perp} + \sin\theta\ (\underline{r} \times \underline{u})$

$\underline{v} = (\underline{r} \otimes \underline{r})\underline{u} + \cos\theta\,(\underline{\underline{I}} - \underline{r} \otimes \underline{r})\,\underline{u} + \sin\theta\ (\underline{r} \times \underline{u})$

Write as $\quad \underline{v} = \underline{\underline{Q}}(\underline{r},\theta)\,\underline{u} \qquad \underline{\underline{A}}\,\underline{u} = A_{ij}u_j$

Need to express: $\quad \underline{w} = \underline{r} \times \underline{u} = \underline{\underline{R}}\,\underline{u}$

$$w_k = \epsilon_{ijk}\, r_i\, u_j = \underbrace{(\epsilon_{ijk}\, r_i)}_{\tilde{R}_{jk}}\, u_j = \tilde{R}_{jk}\, u_j$$

$$= -\underbrace{(\epsilon_{ikj}\, r_i)}_{R_{kj}} = R_{kj}\, u_j$$

$$\boxed{\underline{\underline{R}} = \epsilon_{ikj}\, r_i\, \underline{e}_k \otimes \underline{e}_j = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix}}$$

without minus sign

$\mathrm{tr}(\underline{\underline{R}}) = 0 \qquad \underline{\underline{R}} = -\underline{\underline{R}}^T \Rightarrow$ skew symmetric

$\underline{\underline{R}}$ is the axial tensor of $\underline{r}$

$\underline{r}$ is the axial vector of $\underline{\underline{R}}$

substitute

$$\underline{v} = (\underline{r}\otimes\underline{r})\,\underline{u} + \cos\theta\,(\underline{\underline{I}} - \underline{r}\otimes\underline{r})\,u - \sin\theta\,\underline{\underline{R}}\,\underline{u}$$

$$\underline{v} = \underbrace{\left[(\underline{r} \otimes \underline{r}) + \cos\theta \, (\underline{\underline{I}} - \underline{r} \otimes \underline{r}) - \sin\theta \, \underline{\underline{R}} \right]}_{\underline{\underline{Q}}(\underline{r}, \theta)} \underline{u}$$

Euler representation of finite rotation tensor

$$\boxed{\underline{\underline{Q}}(\underline{r}, \theta) = (\underline{r} \otimes \underline{r}) + \cos\theta \, (\underline{\underline{I}} - \underline{r} \otimes \underline{r}) - \sin\theta \, \underline{\underline{R}}}$$

Infinitesimal rotations:

$$\lim_{\theta \to 0} \underline{\underline{Q}}(r, \theta) = \cancel{(\underline{r} \otimes \underline{r})} + \cancelto{1}{\cos\theta} \, (\underline{\underline{I}} - \cancel{\underline{r} \otimes \underline{r}}) - \sin\theta \, R$$

$$\simeq \underline{\underline{I}} - \sin\theta \, \underline{\underline{R}}$$

$\Rightarrow$ Axial tensor / crossproduct give infinitesimal rotations.

Given $\underline{\underline{Q}}$ what is the angle?

$$Q_{ij} = r_i r_j + \cos\theta \, (\delta_{ij} - r_i r_j) - \sin\theta \, \epsilon_{ikj} \, r_k$$

$$\mathrm{tr}\,(\underline{\underline{Q}}) = Q_{ii} = \underbrace{r_i r_i}_{1} + \cos\theta \, (\underbrace{\delta_{ii}}_{3} - \underbrace{r_i r_i}_{1}) - \cancelto{0}{\sin\theta \, R_{ii}}$$

$$\mathrm{tr}(\underline{\underline{Q}}) = 1 + 2\cos\theta$$

$$\boxed{\cos\theta = \frac{\mathrm{tr}(\underline{\underline{Q}}) - 1}{2}}$$

$\Rightarrow$ axis of rotation can be found similarly $\rightarrow$ more later

Example: rotation around $\underline{e}_3$

$$\underline{\underline{Q}}(\underline{e}_3, \theta) = (\underline{e}_3 \otimes \underline{e}_3) + \cos\theta\,(\underline{\underline{I}} - \underline{e}_3 \otimes \underline{e}_3) - \sin\theta\, \underline{\underline{E}}_3$$

$$[\underline{a} \otimes \underline{b}]_{ij} = a_i b_j \qquad \underline{\underline{E}}_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -0 \\ -0 & 0 & 0 \end{bmatrix} \qquad \underline{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \cos\theta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \\ 0 & 0 & 0 \end{bmatrix} - \sin\theta \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\underline{Q}}(\underline{e}_3, \theta) = \begin{bmatrix} \cos\theta & +\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Finishing some basic tensor algebra

Tensor scalar product (contraction)

analogous to scalar product of vectors

$$\boxed{\underline{\underline{A}} : \underline{\underline{B}} = \mathrm{tr}(\underline{\underline{A}}^T \underline{\underline{B}}) = A_{ij} B_{ij}}$$ scalar

explicit

$$\underline{\underline{A}} : \underline{\underline{B}} = \sum_{i=1}^{3} \sum_{j=1}^{3} A_{ij} B_{ij} = A_{11} B_{11} + A_{12} B_{12} + A_{13} B_{13}$$
$$+ \dots + A_{33} B_{33}$$

$$\underline{\underline{A}}\,\underline{\underline{B}} = A_{ij} B_{jk}$$

$$\underline{\underline{A}} : \underline{\underline{B}} = \mathrm{tr}(\underline{\underline{A}}^T \underline{\underline{B}})$$

$$\underline{\underline{A}}^T \underline{\underline{B}} = (A_{ji}\, \underline{e}_i \otimes \underline{e}_j)(B_{kl}\, \underline{e}_k \otimes \underline{e}_l)$$
$$= A_{ji} B_{kl} (\underline{e}_i \otimes \underline{e}_j)(\underline{e}_k \otimes \underline{e}_l)$$
$$= A_{ij} B_{kl} \underbrace{(\underline{e}_j \cdot \underline{e}_k)}_{\delta_{jk}}\, \underline{e}_i \otimes \underline{e}_l = A_{ji} B_{kl}\, \underline{\underline{\delta_{jk}}}\, \underline{e}_i \otimes \underline{e}_l$$

$$\underline{\underline{A}}^T \underline{\underline{B}} = A_{ji} B_{jl}\, \underline{e}_i \otimes \underline{e}_l$$

$$\mathrm{tr}(\underline{\underline{A}}^T \underline{\underline{B}}) = A_{ji} B_{jl} \delta_{il} = A_{ji} B_{ji} = A_{ij} B_{ij}$$

Common norm for tensor

$$\boxed{|\underline{\underline{A}}| = \sqrt{\underline{\underline{A}} : \underline{\underline{A}}} = \sqrt{A_{ij} A_{ij}} \geq 0}$$

⇒ express the work done during deformation related to shear heating in glaciers