Lecture 8: Change in basis & spectral decomposition

Logistics: - HW 2 is due (3/7) please submit!

- HW 3 will be posted

Last time: - Orthogonal matrices

$$\boxed{\underline{\underline{Q}}\,\underline{a} \cdot \underline{\underline{Q}}\,\underline{b} = \underline{a} \cdot \underline{b}}$$

$\Rightarrow$ preserve magnitude and angle

- Properties: $\boxed{\underline{\underline{Q}}^T = \underline{\underline{Q}}^{-1}}$

$\underline{\underline{Q}}^T \underline{\underline{Q}} = \underline{\underline{I}}$

$\boxed{\det(\underline{\underline{Q}}) = \pm 1}$

$1 \Rightarrow$ rotation $\quad -1 \Rightarrow$ reflection

- Euler representation of finite rotation

$$\underline{\underline{Q}}(\underline{r}, \theta) = \underline{r} \otimes \underline{r} + \cos\theta\,(\underline{\underline{I}} - \underline{r} \otimes \underline{r}) + \sin\theta\,\underline{\underline{R}}$$

$$Q_{ij}(\underline{r}, \theta) = r_i r_j + \cos\theta\,(\delta_{ij} - r_{ij}) + \sin\theta\,\epsilon_{ikj} r_k$$

Today: - find angle and axis of rotation matrix

- Change in basis
- Spectral decomposition

Change in basis

$\underline{v} \in \mathcal{V}$ $\underline{\underline{S}} \in \mathcal{V}^2$ are invariant with change of basis. but their representation $[\underline{v}]$ and $[\underline{\underline{S}}]$ change

Two frames $\{\underline{e}_i\}$ and $\{\underline{e}_i'\}$

$[\underline{a}] = (\underbrace{\underline{a}\cdot\underline{e}_i}_{a_i})\,\underline{e}_i$ $[\underline{a}]' = (\underbrace{\underline{a}\cdot\underline{e}_i'}_{a_i'})\,\underline{e}_i'$

Representation of $\underline{e}_j'$ in $\{\underline{e}_i\}$

$\underline{e}_j' = (\underline{e}_j'\cdot\underline{e}_1)\,\underline{e}_1 + (\underline{e}_j'\cdot\underline{e}_2)\,\underline{e}_2 + (\underline{e}_j'\cdot\underline{e}_3)\,\underline{e}_3$

$= (\underline{e}_j'\cdot\underline{e}_i)\,\underline{e}_i$ i = dummy j = free

$\boxed{\underline{e}_j' = A_{ij}\,\underline{e}_i}$

↑ note transpos

$$\boxed{\underline{\underline{A}} = A_{ij}\,\underline{e}_i \otimes \underline{e}_j \qquad A_{ij} = \underline{e}_i\cdot\underline{e}_j'}$$

change of basis tensor

Similarly express $\underline{e}_i$ in $\{\underline{e}_k'\}$

$\Rightarrow \boxed{\underline{e}_i = A_{ik}\,\underline{e}_k'}$

What type of tensor is $\underline{\underline{A}}$?

$\underline{e}_j' = A_{ij}\, \underline{e}_i \qquad \underline{e}_i = A_{ik}\, \underline{e}_k'$

$\underline{e}_j' = \underbrace{A_{ij} A_{ik}}_{\delta_{jk}} \underline{e}_k' \qquad AB = \underline{A_{ij} B_{jk}}$

because $\{\underline{e}_j'\}$

$A_{ij} A_{ik} = \delta_{jk}$

$\underline{\underline{A}}^T \underline{\underline{A}} = \underline{\underline{I}}$

similarly $\quad \underline{\underline{A}}\,\underline{\underline{A}}^T = \underline{\underline{I}}$

$$\boxed{\underline{\underline{A}}^T \underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{A}}^T = \underline{\underline{I}}}$$

$\underline{\underline{A}}$ is orthogonal

If $\{\underline{e}_i\}$ and $\{\underline{e}_i'\}$ are right handed $\Rightarrow A =$ rotation

$\Rightarrow \det(\underline{\underline{A}}) = 1$

Change in representation

$\underline{v}$ $\underline{\underline{S}}$

$[\underline{v}]$ and $[\underline{\underline{S}}]$ $\{\underline{e}_i\}$

$[\underline{v}']$ and $[\underline{\underline{S}}]'$ $\{\underline{e}_j'\}$

if $\underline{e}_i \neq \underline{e}_i'$ $\Rightarrow$ $[\underline{v}] \neq [\underline{v}]'$ $[\underline{\underline{S}}] \neq [\underline{\underline{S}}]'$

$$\boxed{[\underline{v}] = [\underline{\underline{A}}][\underline{v}]' \qquad [\underline{v}]' = [\underline{\underline{A}}]^T [\underline{v}]}$$

$\underline{v} = v_i \underline{e}_i = v_j' \underline{e}_j'$ where $\underline{e}_j' = A_{ij} \underline{e}_i$

$v_i \underline{e}_i = v_j' A_{ij} \underline{e}_i$

$[\underline{v}] = A_{ij} v_j' \underline{e}_i = [\underline{\underline{A}}][\underline{v}]'$

Similarly:

$$\boxed{\begin{aligned} [\underline{\underline{S}}] &= [\underline{\underline{A}}][\underline{\underline{S}}]'[\underline{\underline{A}}]^T \\ [\underline{\underline{S}}]' &= [\underline{\underline{A}}]^T[\underline{\underline{S}}][\underline{\underline{A}}] \end{aligned}}$$

Invariance of trace

$[\underline{\underline{S}}]$ in $\{\underline{e}_i\}$ and $[\underline{\underline{S}}]'$ in $\{\underline{e}'_i\}$

$$\boxed{\mathrm{tr}[\underline{\underline{S}}] = \mathrm{tr}[\underline{\underline{S}}]'}$$

$$[\underline{\underline{S}}] = [\underline{\underline{A}}][\underline{\underline{S}}]'[\underline{\underline{A}}]^T \quad \text{or} \quad [\underline{\underline{S}}]_{ij} = [\underline{\underline{A}}]_{ik}[\underline{\underline{S}}]'_{kl}[\underline{\underline{A}}]_{jl}$$

$$\begin{aligned}
\mathrm{tr}[\underline{\underline{S}}] = [\underline{\underline{S}}]_{ii} &= [\underline{\underline{A}}]_{ik}[\underline{\underline{S}}]'_{kl}[\underline{\underline{A}}]_{il} \\
&= [\underline{\underline{S}}]'_{kl}\underbrace{[\underline{\underline{A}}]_{ik}[\underline{\underline{A}}]_{il}}_{\delta_{kl}} \\
&= [\underline{\underline{S}}]'_{kk} = \mathrm{tr}[\underline{\underline{S}}]'
\end{aligned}$$

$\Rightarrow$ trace is invariant under change in basis

$\Rightarrow$ good candidate for constitutive laws

Invariance of determinant

$$\boxed{\det[\underline{\underline{S}}] = \det[\underline{\underline{S}}]'}$$

Side: Determinant & Inverse

$$\boxed{\det(\underline{\underline{A}}) = \det\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = \epsilon_{ijk}\,[\underline{\underline{A}}]_{i1}\,[\underline{\underline{A}}]_{j2}\,[\underline{\underline{A}}]_{k3}}$$

$[A]_{i1}$

Properties of determinants:

$$\det(\underline{\underline{A}}\,\underline{\underline{B}}) = \det(A)\det(B)$$

$$\det(\underline{\underline{A}}^T) = \det(\underline{\underline{A}})$$

$$\det(\alpha\underline{\underline{A}}) = \alpha^n \det(\underline{\underline{A}}) \qquad \underline{\underline{A}} \text{ } n\times n \text{ matrix}$$

$\underline{\underline{A}}$ is singular if $\det\underline{\underline{A}} = 0$

if $\det A \neq 0$ then inverse $\underline{\underline{A}}^{-1}$ exist

$$\boxed{\underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \underline{\underline{I}}}$$

Properties: $(\underline{\underline{A}}\,\underline{\underline{B}})^{-1} = \underline{\underline{B}}^{-1}\underline{\underline{A}}^{-1}$

$$(\underline{\underline{A}}^{-1})^{-1} = \underline{\underline{A}}$$

$$(\underline{\underline{A}}^{-1})^T = (\underline{\underline{A}}^T)^{-1}$$

$$(\alpha\underline{\underline{A}})^{-1} = \frac{1}{\alpha}\underline{\underline{A}}^{-1}$$

$$\det(\underline{\underline{A}}^{-1}) = \det(\underline{\underline{A}})^{-1} = \frac{1}{\det(A)}$$

Eigenvalues & Eigen vectors of Tensors

$$\boxed{\underline{\underline{S}}\,\underline{v} = \lambda \underline{\underline{I}}\,\underline{v}}$$

λ = eigen value $\qquad$ $\underline{v}$ = eigen vector

λ's roots of char. polynomial

$$p(\lambda) = \det(\underline{\underline{S}} - \lambda \underline{\underline{I}}) = 0$$

For λ_p we have one or more $\underline{v}_p$

$$(\underline{\underline{S}} - \lambda_p \underline{\underline{I}})\,\underline{v}_p = \underline{0}$$

In CM we are mostly interested in sym. tensors.

Eigenproblem for symmetric tensors

1) All λ_p real
2) All λ_p are positive ($\underline{\underline{S}}$ sym. pos. def.)
3) All $\underline{v}_p$ correspond to distinct λ_p are orthogonal

$\underline{\underline{S}}$ is SPD (sym. pos. def.)

if $\underline{v} \cdot \underline{\underline{S}}\, \underline{v} > 0$ for all $\underline{v} \in \mathcal{V}$

by def. eigen pair: $\underline{\underline{S}}\, \underline{v} = \lambda \underline{v}$

$$\underline{v} \cdot \underline{\underline{S}}\, \underline{v} = \underline{v} \cdot \lambda \underline{v} = \lambda \underbrace{\underline{v} \cdot \underline{v}}_{|\underline{v}|^2} > 0 \qquad \lambda > 0$$

Orthogonality of $\underline{v}_p$'s :

$(\lambda, \underline{v})$ and $(\omega, \underline{u})$ $\boxed{\lambda \neq \omega}$

$\underline{\underline{S}}\, \underline{v} = \lambda \underline{v}$ $\qquad$ $\underline{\underline{S}}\, \underline{u} = \omega \underline{u}$

Consider: $\lambda (\underline{v} \cdot \underline{u}) = (\lambda \underline{v} \cdot \underline{u}) =$

$$= (\underline{\underline{S}}\, \underline{v} \cdot \underline{u}) = (\underline{v} \cdot \underline{\underline{S}}^T \underline{u}) \qquad \underline{\underline{S}} = \underline{\underline{S}}^T$$

$$= (\underline{v} \cdot \underline{\underline{S}}\, \underline{u}) = (\underline{v} \cdot \omega \underline{u})$$

$$\lambda \underbrace{(\underline{v} \cdot \underline{u})}_{0} = \omega \underbrace{(\underline{v} \cdot \underline{u})}_{0}$$

$\Rightarrow \quad \underline{v} \cdot \underline{u} = 0$

$\Rightarrow$ use orthogonal $\underline{v}_p$'s as frame $\{\underline{v}_p\}$

Spectral decomposition

If $\underline{\underline{S}} = \underline{\underline{S}}^T$ there exists a frame $\{\underline{v}_p\}$

$$\boxed{\underline{\underline{S}} = \sum_{i=1}^{3} \lambda_i \; \underline{v}_i \otimes \underline{v}_i}$$

$$\underline{I} = \underline{v}_i \otimes \underline{v}_i$$

$$\underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{I}} = \underline{\underline{A}}\,(\underline{v}_i \otimes \underline{v}_i) = (\underline{\underline{A}}\,\underline{v}_i \otimes \underline{v}_i)$$

$$= (\underline{\underline{A}}\,\underline{v}_i) \otimes \underline{v}_i = \sum_{i=1}^{3} (\lambda_i \underline{v}_i) \otimes \underline{v}_i \, .$$

$\underline{\underline{A}}\,(\underline{u} \otimes \underline{v}) = (\underline{\underline{A}}\,\underline{u} \otimes \underline{v}) \Rightarrow$ HW3

Representation in eigenframe

$$[\underline{\underline{S}}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

diagonal tensor

The principal invariants of $\underline{\underline{S}}$

$$\boxed{\begin{aligned} I_1(\underline{\underline{S}}) &= tr(\underline{\underline{S}}) = \lambda_1 + \lambda_2 + \lambda_3 \\ I_2(\underline{\underline{S}}) &= \tfrac{1}{2}\left((tr(\underline{\underline{S}}))^2 - tr(\underline{\underline{S}}^2)\right) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 \\ I_3(\underline{\underline{S}}) &= det(\underline{\underline{S}}) = \lambda_1 \lambda_2 \lambda_3 \end{aligned}}$$

These 3 scalars are frame invariant

$$I_S = \{ I_i(\underline{\underline{S}}) \}$$

Examples of invariant use:

$I_1(\underline{\underline{S}})$ = pressure / mean normal stress

I_2 is important in theories of creep

I_2 & I_3 important in theories of plastic yield

Char. polynomial in terms of invariants

$$det(\underline{\underline{S}} - \lambda \underline{\underline{I}}) = -\lambda^3 + I_1(\underline{\underline{S}})\lambda^2 - I_2(\underline{\underline{S}})\lambda + I_3(\underline{\underline{S}}) = 0$$

Tensor square root

$\underline{\underline{U}} = \sqrt{\underline{\underline{C}}}$

$$\boxed{\underline{\underline{U}} = \sum_{i=1}^{3} \sqrt{\lambda_i}\, \underline{e}_i \otimes \underline{e}_i}$$