Lecture 9: Principal Stresses

Logistics: - HW3 due Th

- HW2 only 4 submissions!

For this class HWs are essential

Last time: Change of basis $\{\underline{e}\}$ and $\{\underline{e}'\}$

Representation $[\underline{v}]$ vs. $[\underline{v}]'$

$[\underline{\underline{S}}]$ vs. $[\underline{\underline{S}}]'$

Change in basis tensor: $\underline{\underline{A}}$ $\boxed{A_{ij} = \underline{e}_i \cdot \underline{e}_j'}$

$[\underline{v}] = [\underline{\underline{A}}][\underline{v}]'$ $\qquad [\underline{\underline{S}}] = [\underline{\underline{A}}][\underline{\underline{S}}]'[\underline{\underline{A}}]^T$

$[\underline{v}]' = [\underline{\underline{A}}]^T[\underline{v}]$ $\qquad [\underline{\underline{S}}]' = [\underline{\underline{A}}]^T[\underline{\underline{S}}][\underline{\underline{A}}]$

Spectral decomposition (sym $\underline{\underline{S}} = \underline{\underline{S}}^T$)

$$\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} = \boxed{\underline{\underline{S}} = \sum_{i=1}^{3} \lambda_i \, \underline{v}_i \otimes \underline{v}_i} \qquad \underline{\underline{S}}\,\underline{v} = \lambda\,\underline{v}$$

Principal invariants:

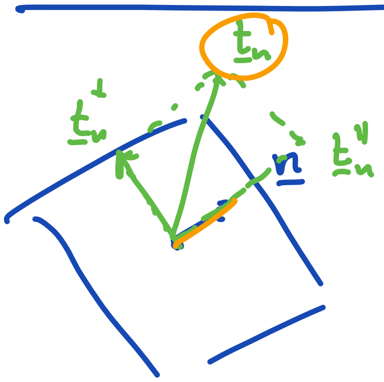
$$I_1 = \mathrm{tr}(\underline{\underline{S}}) = \lambda_1 + \lambda_2 + \lambda_3$$

$$I_2 = \tfrac{1}{2}\left(\mathrm{tr}(\underline{\underline{S}})^2 + \mathrm{tr}(\underline{\underline{S}}^2)\right) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$$

$$I_3 = \det(\underline{\underline{S}}) = \lambda_1\lambda_2\lambda_3$$

Today: Principal stresses

Normal & Shear Stresses



$$\underline{t}_n = \underline{\underline{\sigma}}\,\underline{n}$$

$$\underline{t}_n^{\parallel} = \underline{\underline{P}}^{\parallel}\,\underline{t}_n = (\underline{n}\otimes\underline{n})\,\underline{t}_n$$

$$\underline{t}_n^{\perp} = \underline{\underline{P}}^{\perp}\,\underline{t}_n = (\underline{\underline{I}}-\underline{n}\otimes\underline{n})\,\underline{t}_n = \underline{m}\otimes\underline{m}$$

$|n|\ |t_n|$ s

normal stress: $\sigma_n = \underline{n}\cdot\underline{t}_n = \underline{n}\cdot\underline{\underline{\sigma}}\,n = n_i\,\sigma_{ij}\,n_j$

shear stress: $\tau = \underline{m}\cdot\underline{t}_n = \underline{m}\cdot\underline{\underline{\sigma}}\,\underline{n} = m_i\,\sigma_{ij}\,n_j$

Extremal Stress Values

I) Max and Min Normal Stresses

Q: Given a $\underline{\underline{\sigma}}$ at $\underline{x}$ what are the $\underline{n}$'s corresponding to to max & min normal stress.

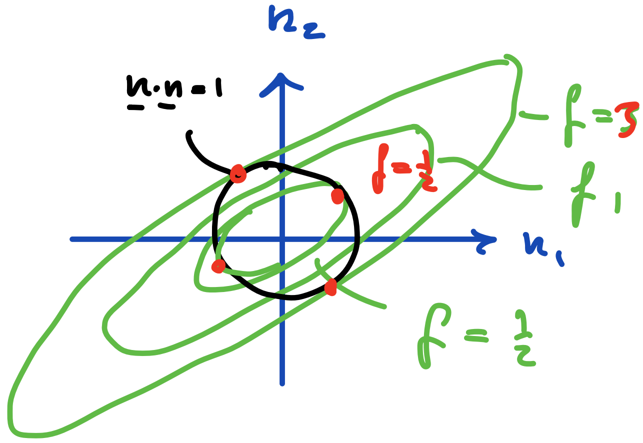
$\Rightarrow$ Optimization problem for $\underline{n}$ with constraint that $|\underline{n}| = 1$

Lagrange multiplier method

$\underline{n}\cdot\underline{n} - 1 = 0$

$$\boxed{\begin{aligned}\mathcal{L}(\underline{n},\lambda) &= \underline{n}\cdot\underline{\sigma}\,\underline{n} - \lambda(\underline{n}\cdot\underline{n}-1)\\ \mathcal{L}(n_i,\lambda) &= \underbrace{n_i\,\sigma_{ij}\,n_j}_{\text{function}} - \lambda\underbrace{(n_i n_i - 1)}_{\text{constraint}}\end{aligned}}$$

Lagrange multiplier



$f(\underline{n}) = \underline{n}\cdot\underline{\sigma}\,\underline{n}$

is quadratic in components of n

$n = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$

Extremal values are stationary points of $\mathcal{L}$

$$\frac{\partial\mathcal{L}}{\partial\lambda} = n_i n_i - 1 = 0$$

$$\frac{\partial\mathcal{L}}{\partial n_k} = \sigma_{ij}\frac{\partial}{\partial n_k}\left[(n_i\;n_j)\right] - \frac{\partial}{\partial n_k}\lambda\left[(n_i n_i - 1)\right]$$

$$\frac{\partial n_1}{\partial n_1} = 1 \qquad \frac{\partial n_1}{\partial n_2} = 0$$

$$\frac{\partial n_i}{\partial n_k} = n_{i,k} = \delta_{ik}$$

$$= \sigma_{ij}\left[n_{i,k}\,n_j + n_i\,n_{j,k}\right] - \lambda\,2\,n_{i,k}\,n_i$$

$$= \sigma_{ij} [\delta_{ik} n_j + \delta_{jk} n_i] - 2\lambda \, \delta_{ik} n_i$$

$$= \sigma_{ij} \delta_{ik} n_j + \sigma_{ij} \delta_{jk} n_i - 2\lambda \, \delta_{ik} n_i$$

$$= \sigma_{kj} n_j + \sigma_{ik} n_i - 2\lambda \, n_k$$

use $\sigma_{ik} = \sigma_{ki}$

$$= \underline{\cancel{2} (\sigma_{kj} n_j - \lambda n_k)} = 0$$

In dyadic notation: $\boxed{(\underline{\underline{\sigma}} - \lambda \underline{\underline{I}}) \underline{n} = 0}$ $\quad |\underline{n}| = 1$

Lagrange multiplies method leads to eigen value problem

What is significance of λ?

$$\underline{n} \cdot (\underline{\underline{\sigma}} - \lambda \underline{\underline{I}}) \underline{n} = 0$$

$$\underbrace{\underline{n} \cdot \underline{\underline{\sigma}} \underline{n}}_{\sigma_n} - \lambda \underbrace{\underline{n} \cdot \underline{n}}_{1} = 0 \quad \Rightarrow \quad \boxed{\sigma_n = \lambda}$$

Solve eigenvalue problem to find extremal normal stresses

λ_i's principal normal stresses $\Rightarrow \lambda_i = \sigma_i$

$\underline{n}_i$'s principal directions of $\underline{\underline{\sigma}}$

$$\boxed{\underline{\underline{\sigma}} = \sum_{i=1}^{3} \sigma_i \; \underline{n}_i \otimes \underline{n}_i}$$ eigen frame

$$[\underline{\underline{\sigma}}] = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix}$$

tractions

$$\underline{t}_{n_i} = \underline{\underline{\sigma}}\, \underline{n}_i = \underbrace{\sum_{i=1}^{3} \sigma_i \, (\underline{n}_i \otimes \underline{n}_i)}_{\sigma} \, (\cancel{e_1} \quad \underline{n}_j)$$

$\underline{t}_{n_i} = \sigma_i \, \underline{n}_i$ $\qquad \underline{t}_{n_i} \parallel \underline{n}_i$

$\Rightarrow$ no shear stress

II) Max & min shear stresses.

look in $\{\underline{n}_i\}$ principal stresse

$\underline{s} = s_i \underline{n}_i \qquad\qquad s_i = \underline{s}\cdot\underline{n}_i$

traction vector ~~in~~ on plane given by $\underline{s}$

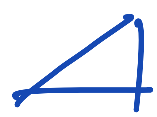
$$\underline{t}_s = \underline{\underline{\sigma}}\,\underline{s} = \sum_{i=1}^{3} \sigma_i\, n_i \otimes n_i \,(s_j \underline{n}_j)$$

$$= \sum_{i=1}^{3} \sigma_i\, s_i\, (n_i \otimes n_i)\, \underline{n}_j$$

$$= \sum \sigma_i\, s_i\, (\underbrace{\underline{n}_i \cdot n_j}_{\delta_{ij}})\, n_j$$

$$\underline{t}_s = \sum_{i=1}^{3} \sigma_i\, s_i\, \underline{n}_i$$

$$\underline{t}_s = \sigma_1 s_1 \underline{n}_1 + \sigma_2 s_2 n_2 + \sigma_3 s_3 n_3$$



mag. of normal & shear shear

$$\sigma_n = \underline{s}\cdot\underline{t}_s = \sigma_1 s_1^2 + \sigma_2 s_2^2 + \sigma_3 s_3^2$$

$$\underline{\tau}^2 = |\underline{t}_s|^2 - \sigma_n^2$$

$$\tau^2 = \sigma_1^2 s_1^2 + \sigma_2^2 s_2^2 + \sigma_3^2 s_3^2 - (\sigma_1 s_1^2 + \sigma_2 s_2^2 + \sigma_3 s_3^2)^2$$

In index notation

$$\boxed{\tau^2 = \sum_{i=1}^{3} \sigma_i^2 s_i^2 - \left(\sum_{i=1}^{3} \sigma_i s_i^2\right)^2}$$

function we are optimizing under the constraint $|\underline{s}|^2 = 1$ $\quad s_1^2 + s_2^2 + s_3^2 = 1$

solve using direct substitution

I) Eliminate $s_3^2 = 1 - s_1^2 - s_2^2 \Rightarrow \tau^2 = \tau^2(s_1, s_2)$

$$\tau^2 = \sigma_1^2 s_1^2 + \sigma_2^2 s_2^2 + \sigma_3^2 s_3^2 - (\sigma_1 s_1^2 + \sigma_2 s_2^2 + \sigma_3 s_3^2)^2$$

$$= \sigma_1^2 s_1^2 + \sigma_2^2 s_2^2 + \sigma_3^2 (1 - s_1^2 - s_2^2) - \left(\sigma_1 s_1^2 + \sigma_2 s_2^2 + \sigma_3 (1 - s_1^2 - s_2^2)\right)^2$$

$\Rightarrow$ constraint is incorporated

we just need to find $\frac{\partial \tau^2}{\partial s_1} = \frac{\partial \tau^2}{\partial s_2} = 0$

$$\frac{\partial \tau^2}{\partial s_1} = 2 s_1 (\sigma_1 - \sigma_3) \left\{\sigma_1 - \sigma_3 - 2\left[(\sigma_1 - \sigma_3) s_1^2 + (\sigma_2 - \sigma_3) s_2^2\right]\right\} = 0$$

$$\frac{\partial \tau^2}{\partial s_2} = 2 s_2 (\sigma_2 - \sigma_3) \left\{\sigma_2 - \sigma_3 - 2\left[(\sigma_1 - \sigma_3) s_1^2 + (\sigma_2 - \sigma_3) s_2^2\right]\right\} = 0$$

First solution: $s_1 = s_2 = 0 \Rightarrow s_3 = \pm 1 \qquad \underline{s} = \pm \underline{n}_3$

$$\tau^2 = \sigma_3^2 \cdot 1 - (\sigma_3 \cdot 1)^2 = 0$$

$\Rightarrow$ minimum shear stress in principal plane

Second solution: $s_1 = 0$

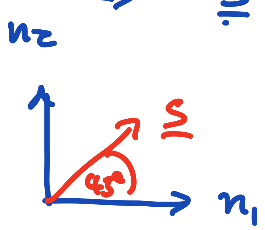
$$\frac{\partial \tau^2}{\partial s_2} = \sigma_2 - \sigma_3 - 2\left[(\sigma_2 - \sigma_3)\, s_2^2\right] = 0$$

$$\cancel{(\sigma_2 - \sigma_3)}\,(1 - 2 s_2^2) = 0 \qquad s_2^2 = \frac{1}{2}$$

$$\Rightarrow \sigma_2 = \pm \frac{1}{\sqrt{2}}$$

from $\quad s_2^2 + s_3^2 = 1 \qquad s_3 = \pm \frac{1}{\sqrt{2}}$

$$\Rightarrow \underline{s} = \pm \frac{1}{\sqrt{2}} \underline{n}_2 \pm \frac{1}{\sqrt{2}} \underline{n}_3$$



$$\tau^2 = \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} - \left(\frac{\sigma_2}{2} + \frac{\sigma_3}{2}\right)^2$$

$$\tau^2 = \left(\frac{\sigma_2 - \sigma_3}{2}\right)^2$$

Two solutions

min $\quad \tau = 0 \qquad \underline{s} = \pm \underline{n}_3$

max $\quad \tau = \frac{1}{2}(\sigma_2 - \sigma_3) \qquad \underline{s} = \pm \frac{\underline{n}_2}{\sqrt{2}} \pm \frac{\underline{n}_3}{\sqrt{2}}$

Playing this two more times

Min. shear stresses

$\tau = 0$ on $\quad \underline{n} = \pm \underline{n}_1 \quad \underline{n} = \pm \underline{n}_2 \quad \underline{n} = \pm \underline{n}_3$

Max shear stresses

$\tau_{23} = \frac{1}{2}(\sigma_2 - \sigma_3)$ on $\underline{n}_{23} = \frac{1}{\sqrt{2}}(\pm \underline{n}_2 \pm \underline{n}_3)$

$\tau_{13} = \frac{1}{2}(\sigma_1 - \sigma_3)$ on $\underline{n}_{13} = \frac{1}{\sqrt{2}}(\pm \underline{n}_1 \pm \underline{n}_3)$

$\tau_{12} = \frac{1}{2}(\sigma_1 - \sigma_2)$ on $\underline{n}_{12} = \frac{1}{\sqrt{2}}(\pm \underline{n}_1 \pm \underline{n}_2)$