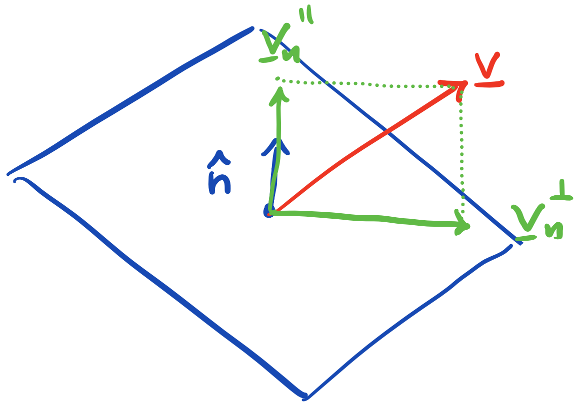
Projection & Reflection tensors

commonly used to partition forces on a surface.



$$\underline{v} = \underline{v}_n^{\parallel} + \underline{v}_n^{\perp}$$

Use dot product:

$$\underline{v}_n^{\parallel} = (\underline{v} \cdot \hat{\underline{n}})\, \hat{\underline{n}}$$

$$\underline{v}_n^{\perp} = \underline{v} - \underline{v}_n^{\parallel}$$

Tensors? $\underline{v}^{\parallel} = \underline{\underline{P}}^{\parallel}\, \underline{v}$

$\underline{v}^{\perp} = \underline{\underline{P}}^{\perp}\, \underline{v}$

use dyadic property!

$$\underline{v}_n^{\parallel} = (\underline{v} \cdot \hat{\underline{n}})\, \hat{\underline{n}} = (\hat{\underline{n}} \otimes \hat{\underline{n}})\, \underline{v} = \underline{\underline{P}}_n^{\parallel}\, \underline{v}$$

$$\underline{v}_n^{\perp} = \underline{v} - (\hat{\underline{n}} \otimes \hat{\underline{n}})\, \underline{v} = (\underline{\underline{I}} - \hat{\underline{n}} \otimes \hat{\underline{n}})\, \underline{v} = \underline{\underline{P}}_n^{\perp}\, \underline{v}$$

Projection tensors:

$$\boxed{\underline{\underline{P}}_n^{\parallel} = \hat{\underline{n}} \otimes \hat{\underline{n}}} \qquad \boxed{\underline{\underline{P}}_n^{\perp} = \underline{\underline{I}} - \hat{\underline{n}} \otimes \hat{\underline{n}}}$$

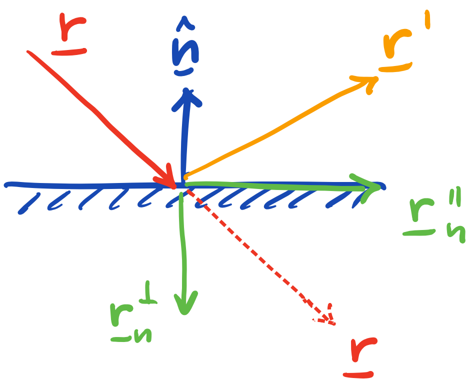
Properties: $\underline{\underline{P}} = \underline{\underline{P}}^T$ symmetric (HW2)

$$\underline{\underline{P}}^2 = \underline{\underline{P}}$$

$$\underline{\underline{P}}^{\parallel} + \underline{\underline{P}}^{\perp} = \underline{\underline{I}}$$

$$\underline{\underline{P}}^{\parallel}\,\underline{\underline{P}}^{\perp} = \underline{\underline{0}}$$

Reflections



incoming: $\underline{r} = \underline{r}_n^{\parallel} + \underline{r}_n^{\perp}$

reflected: $\underline{r}' = \underline{r}_n^{\parallel} - \underline{r}_n^{\perp}$

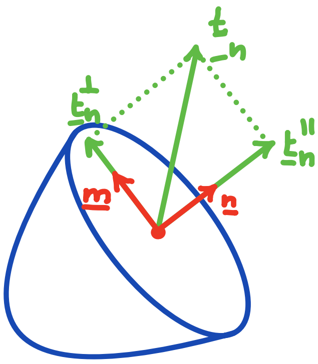
$$\underline{r}' = (\underline{\underline{P}}_n^{\parallel} - \underline{\underline{P}}^{\perp})\,\underline{r}$$

$$\underline{r}' = (\underline{\underline{I}} - 2\,\hat{\underline{n}} \otimes \hat{\underline{n}})\,\underline{r}$$

$$\underline{r}' = \underline{\underline{R}}_n\,\underline{r}$$

Reflection tensor: $\boxed{\underline{\underline{R}}_n = \underline{\underline{I}} - 2\,\hat{\underline{n}} \otimes \hat{\underline{n}}}$

Normal and Shear Stresses



Consider an arbitrary surface in B

projection matrices:

$$\underline{\underline{P}}^{\parallel} = \underline{n} \otimes \underline{n}$$

$$\underline{\underline{P}}^{\perp} = \underline{\underline{I}} - \underline{n} \otimes \underline{n} = \underline{m} \otimes \underline{m}$$

normal stress: $\underline{t}_n^{\parallel} = \underline{\underline{P}}^{\parallel}\, \underline{t}_n = (\underline{n} \cdot \underline{t}_n)\, \underline{n} = \sigma_n \underline{n}$

shear stress: $\underline{t}_n^{\perp} = \underline{\underline{P}}^{\perp}\, \underline{t}_n = (\underline{m} \cdot \underline{t}_n)\, \underline{m} = \tau\, \underline{m}$

The magnitudes of these stresses are:

$$\sigma_n = |\underline{t}_n^{\parallel}| = \underline{n} \cdot \underline{t}_n = \underline{n} \cdot \underline{\underline{\sigma}}\, \underline{n} \quad \text{or} \quad \sigma_n = n_i \sigma_{ij} n_j$$

$$\tau = |\underline{t}_n^{\perp}| = \underline{m} \cdot \underline{t}_n = \underline{m} \cdot \sigma\, \underline{n} \quad \text{or} \quad \tau = m_i \sigma_{ij} n_j$$

normal stresses:

$\sigma_n > 0$ tensile

$\sigma_n < 0$ compressive

From geometry: $\underline{t}_n = \underline{t}_n^{\parallel} + \underline{t}_n^{\perp}$

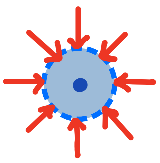
$$|\underline{t}_n|^2 = |\sigma_n \underline{n}|^2 + |\tau \underline{n}|^2 = \sigma_n^2 + \tau_n^2$$

Simple states of stress

I) Hydrostatic stress

$$\boxed{\underline{\underline{\sigma}} = -p\,\underline{\underline{I}}} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$$

$$\Rightarrow \underline{t}_n = \underline{\underline{\sigma}}\,\underline{n} = -p\,\underline{n} \qquad \text{for all } \underline{n}$$



$$\underline{t}_n^{\parallel} = \underline{\underline{P}}_n^{\parallel}\,\underline{t} = (\underline{n}\otimes\underline{n})(-p\,\underline{n}) = -p\,(\underline{n}\cdot\underline{n})\,\underline{n} = -p\,\underline{n}$$

$$\Rightarrow \underline{t}_n = \underline{t}_n^{\parallel} \qquad \underline{t}_n^{\perp} = 0$$

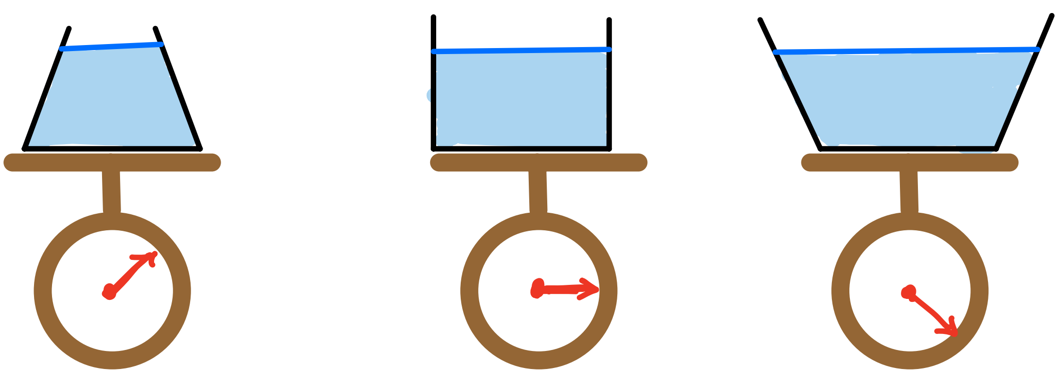
$$\left.\begin{array}{ll} \text{normal stress:} & \sigma_n = -p \\ \text{shear stress:} & \tau = 0 \end{array}\right\} \text{ on all planes}$$

Pascal's law:

The pressure in a fluid at rest is independent of the direction of a surface. Pressure is a scalar!

Hydrostatic paradox: (Blaise Pascal)

Weight different but the force on base is same $f = pA$

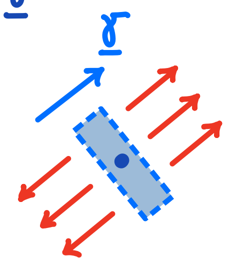


II) Uniaxial stress

$$\boxed{\underline{\underline{\sigma}} = \sigma \; \underline{\gamma} \otimes \underline{\gamma}}$$

$\Rightarrow \underline{t}_n = \underline{\underline{\sigma}}\,\underline{n} = \sigma\,(\underline{\gamma} \cdot \underline{n})\,\underline{\gamma}$

($\underline{\gamma}$ is unit vector)



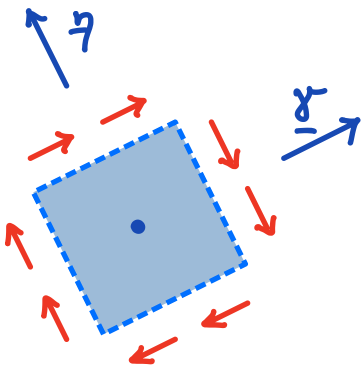
Traction is always parallel to $\underline{\gamma}$ and vanishes on surfaces with normal perpendicular to $\underline{\gamma}$.

$\sigma > 0$: pure tension

$\sigma < 0$: pure compression

III, Pure shear stress $\qquad \underline{\gamma} \cdot \underline{\eta} = 0$

$$\boxed{\underline{\underline{\sigma}} = \tau \left(\underline{\gamma} \otimes \underline{\eta} + \underline{\eta} \otimes \underline{\gamma} \right)} \Rightarrow \underline{t}_n = \underline{\underline{\sigma}}\, \underline{n} = \tau (\underline{\eta} \cdot \underline{n}) \underline{\gamma} + \tau (\underline{\gamma} \cdot \underline{n}) \underline{\eta}$$



$\underline{n} = \underline{\eta}: \quad \underline{t}_n = \tau \underline{\gamma}$

$\underline{n} = \underline{\gamma}: \quad \underline{t}_n = \tau \underline{\eta}$

IV, Plane stress

If there exists a pair of orthogonal vectors $\underline{\gamma}$ and $\underline{\eta}$ such that the matrix representation of $\underline{\underline{\sigma}}$ in the frame $\{\underline{\gamma}, \underline{\eta}, \underline{\gamma} \times \underline{\eta}\}$ is of the form

$$[\underline{\underline{\sigma}}] = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

then a state of plane stress exists.

Q: Is uniaxial stress a plane stress?

$\underline{\underline{\sigma}} = \sigma\, \underline{a} \otimes \underline{a}$

Pick a frame $\{\underline{e}_i\}$ and evaluate $[\underline{\underline{\sigma}}]$.

What frame $\underline{e}_1 = \underline{a}$ know $\underline{e}_2 \cdot \underline{a} = \underline{e}_3 \cdot \underline{a} = \underline{0}$

$\sigma_{ij} = \underline{e}_i \cdot \underline{\underline{\sigma}}\, \underline{e}_j$

substitute with $\underline{a} = \underline{e}_1$

$\sigma_{ij} = \underline{e}_i \cdot (\sigma\, \underline{e}_1 \otimes \underline{e}_1)\, \underline{e}_j$

$= \sigma\, \underline{e}_i \cdot (\underline{e}_1 \cdot \underline{e}_j)\, \underline{e}_1 = \sigma (\underline{e}_i \cdot \underline{e}_1)(\underline{e}_j \cdot \underline{e}_1)$

$\sigma_{11} = \sigma\, (\underline{e}_1 \cdot \underline{e}_1)(\underline{e}_1 \cdot \underline{e}_1) = \sigma$

$\sigma_{12} = \sigma\, (\underline{e}_1 \cdot \underline{e}_1)\cancelto{0}{(\underline{e}_1 \cdot \underline{e}_2)} = 0$

$\sigma_{22} = \sigma\, (\underline{e}_2 \cdot \underline{e}_1)(\underline{e}_1 \cdot \underline{e}_2) = 0$

...

$\Rightarrow \quad [\underline{\underline{\sigma}}] = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ✓ plane stress

Spherical and deviatoric stress tensors

The Cauchy stress tensor can be decomposed as $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}_S + \underline{\underline{\sigma}}_D$

spherical stress tensor: $\underline{\underline{\sigma}}_S = -p\underline{\underline{I}}$ $\quad p = -\frac{1}{3}\mathrm{tr}(\underline{\underline{\sigma}})$

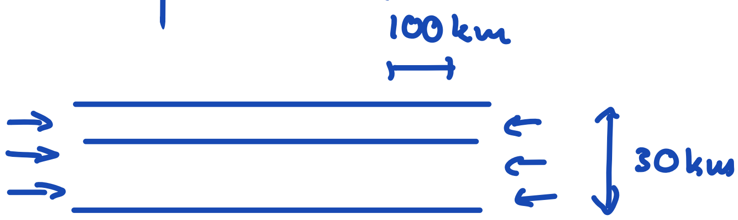
deviatoric stress tensor: $\underline{\underline{\sigma}}_D = \underline{\underline{\sigma}} + p\underline{\underline{I}}$

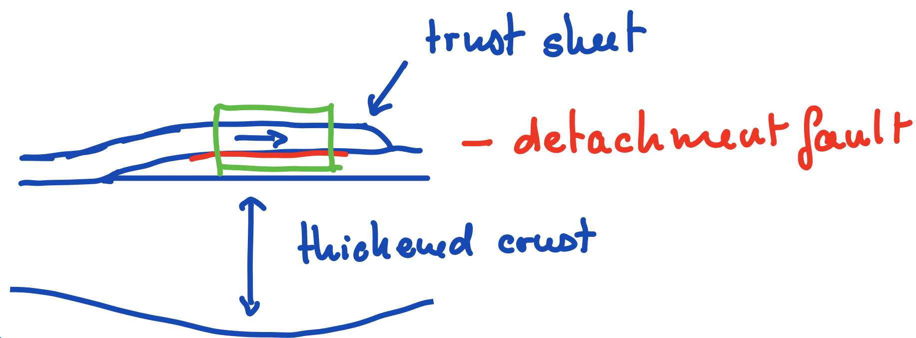
The pressure $p = -\frac{1}{3}\mathrm{tr}(\underline{\underline{\sigma}}) = -\frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$ can be interpreted as the mean normal traction. The spherical stress is the part of $\underline{\underline{\sigma}}$ that changes the volume of the body. Note that $p > 0$ corresponds to compression.

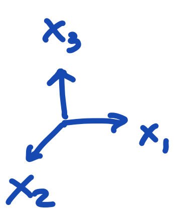
The deviatoric stress is the part of $\underline{\underline{\sigma}}$ that changes the shape of a body without changing its volume. By definition $\mathrm{tr}\,\underline{\underline{\sigma}}_D = 0$.

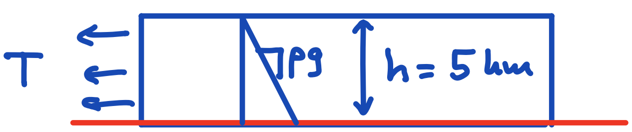
$$\underline{F}_s[\partial B] = -\int_B \rho g \underline{e}_3 \, dV = -\rho g \underline{e}_3 \underbrace{\int_B dV}_{V_B} = -\rho g V_B \underline{e}_3$$

Example: Fault block on detachment









Normal stresses:

Verital stress: $\sigma_{33} = \rho g h$

Horizontal stress (x_1-dir): $\sigma_{11} = \kappa \sigma_{33} - T$

Horizontal stress (x_2 dir): $\sigma_{22} = \kappa \sigma_{33}$

In fluid $\kappa = 1$, but in rock $\kappa < 1$ due to finite strength.

T is tensile tectonic stress

Assume only shear stress is in 1-3 coord. plane

$\sigma_{13} = \sigma_{31} = \mu\,(\rho g h)$ $\quad\quad$ μ = coefficient of friction

$\sigma_{21} = \sigma_{12} = 0 \quad\quad \sigma_{23} = \sigma_{32} = 0$

This results in following stress tensor:

$$\underline{\underline{\sigma}} = \begin{bmatrix} \kappa\rho g h - T & 0 & \mu\rho g h \\ 0 & \kappa\rho g h & 0 \\ \mu\rho g h & 0 & \rho g h \end{bmatrix}$$

Traction on basal plane:

$$\underline{t}(\underline{e}_3) = \underline{\underline{\sigma}}\,\underline{e}_3 = \begin{bmatrix} \mu \\ 0 \\ 1 \end{bmatrix} \rho g h$$

Normal stress of fault: $\underline{t}(\underline{e}_3)\cdot\underline{e}_3 = \rho g h$

Shear stress on fault: $\underline{t}(\underline{e}_3)\cdot\underline{e}_1 = \mu\,\rho g h$

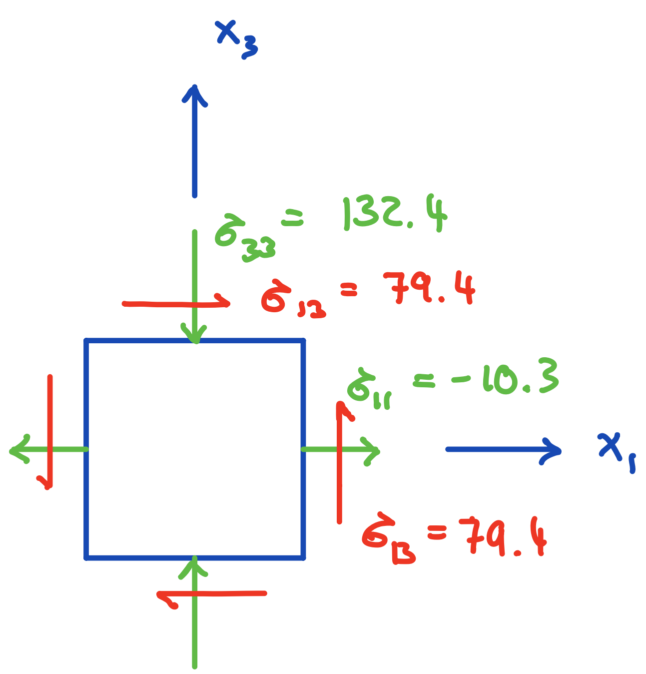
Assume following numbers:

$\rho = 2700 \ kg/m^3$ $\quad h = 5000 \ m$

$g = 9.8 \ m/s^2$ $\quad T = 50 \ MPa$

$K = 0.3$ $\quad \mu = 0.6$

$$\Rightarrow \quad \underline{\underline{\sigma}} = \begin{bmatrix} -10.3 & 0 & 79.4 \\ 0 & 39.7 & 0 \\ 79.4 & 0 & 132.4 \end{bmatrix} \ MPa$$



$$\underline{t}_1 = \underline{t}(\underline{e}_1) = \underline{\underline{\sigma}}\, \underline{e}_1 = \begin{bmatrix} -10.3 \\ 0 \\ 79.4 \end{bmatrix} \ MPa$$

$$\underline{t}_3 = \underline{t}(\underline{e}_3) = \underline{\underline{\sigma}}\, \underline{e}_3 = \begin{bmatrix} 79.4 \\ 0 \\ 132.4 \end{bmatrix} \ MPa$$

Normal & shear stress on x_1-coord. plane:

$$\underline{t}_1^{\parallel} = \underline{\underline{P}}_1^{\parallel}\,\underline{t}_1 = (\underline{e}_1 \otimes \underline{e}_1)\,\underline{t}_1 = (\underbrace{\underline{e}_1 \cdot \underline{t}_1}_{\sigma_n = \sigma_{11}})\,\underline{e}_1 = \begin{bmatrix} -10.3 \\ 0 \\ 0 \end{bmatrix} \text{MPa}$$

$$\underline{t}^{\perp} = \underline{\underline{P}}^{\perp}\,\underline{t}_1 = \underline{t}_1 - \underline{t}_1^{\parallel} = \begin{bmatrix} -10.3 \\ 0 \\ 79.4 \end{bmatrix} - \begin{bmatrix} -10.3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 79.4 \end{bmatrix} \text{MPa}$$

Normal & shear stress on x_3-coordinate plane:

$$\underline{t}_3^{\parallel} = \underline{\underline{P}}_3^{\parallel}\,\underline{t}_3 = (\underline{e}_3 \otimes \underline{e}_3)\,\underline{t}_3 = (\underline{t}_3 \cdot \underline{e}_3)\,\underline{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 132.4 \end{bmatrix} \text{MPa}$$

$$\underline{t}_3^{\perp} = \underline{t}_3 - \underline{t}_3^{\parallel} = \begin{bmatrix} 79.4 \\ 0 \\ 132.4 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 132.4 \end{bmatrix} = \begin{bmatrix} 79.4 \\ 0 \\ 0 \end{bmatrix} \text{MPa}$$