Newtonian Fluids

A fluid is incompressible Newtonian if:

1) Reference mass density uniform: $\rho_0(X) = \rho_0$

2) Fluid is incompressible $\nabla_x \cdot \underline{v} = 0$

3) Cauchy stress field is Newtonian

$$\underline{\underline{\sigma}} = -p\,\underline{\underline{I}} + \mathbb{C}\,\nabla_x \underline{v}$$

Prop 1 + 2 $\Rightarrow$ $\rho(\underline{x},t) = \rho_0 > 0$

Reactive stress: $\underline{\underline{\sigma}}^r = -p\,\underline{\underline{I}}$

p is multiplier for $\nabla_x \cdot \underline{v} = 0$

Active stress: $\underline{\underline{\sigma}}^a = \mathbb{C}\,\nabla_x \underline{v} = 2\mu\,\mathrm{sym}(\nabla_x \underline{v})$

by frame indifference

μ = absolute viscosity

In limit $\mu \to 0$ Newtonian fluid reduces to ideal fluid.

Navier - Stokes Equations

Setting $\rho = \rho_0$ and $\underline{\underline{\sigma}} = -p\underline{\underline{I}} + 2\mu \,\mathrm{sym}(\nabla_x \underline{v})$

we obtain lin. mom. balance

$$\rho_0 \dot{v} = \nabla_x \cdot \left(-p\underline{\underline{I}} + 2\mu \,\mathrm{sym}(\nabla_x \underline{v})\right) + \rho_0 \underline{b}$$

from mat. deriv. $\dot{\underline{v}} = \frac{\partial v}{\partial t} + (\nabla_x \underline{v})\underline{v}$

assuming $\mu = \text{constant}$ we have

$$\nabla \cdot \underline{\underline{\sigma}} = -\nabla_x p + \mu \nabla_x \cdot \nabla_x \underline{v} + \mu \nabla_x \cdot (\nabla_x \underline{v})^T$$

$$\nabla_x \cdot \nabla_x \underline{v} = v_{i,jj} \underline{e}_i = \nabla_x^2 \underline{v}$$

$$\nabla_x \cdot (\nabla_x \underline{v})^T = v_{j,ij} \underline{e}_i = v_{j,ji} \underline{e}_i = \nabla_x (\cancel{\nabla_x \cdot \underline{v}})^{0}$$

$$\Rightarrow \nabla \cdot \underline{\underline{\sigma}} = -\nabla_x p + \mu \nabla_x^2 \underline{v}$$

so that

$$\boxed{\begin{gathered} \rho_0 \left[\frac{\partial v}{\partial t} + (\nabla_x \underline{v})\underline{v}\right] = \mu \nabla_x^2 \underline{v} - \nabla_x p + \rho_0 b \\ \nabla_x \cdot \underline{v} = 0 \end{gathered}}$$

Mechanical energy considerations

Stress power of Newtonian fluid is

$$\underline{\underline{\sigma}} : \underline{\underline{d}} = (-p\underline{\underline{I}} + 2\mu \underline{\underline{d}}) : \underline{\underline{d}} = -p \underbrace{\underline{\underline{I}} : \underline{\underline{d}}}_{\nabla_x \cdot \underline{v} = 0} + 2\mu\, \underline{\underline{d}} : \underline{\underline{d}}$$

$$= 2\mu\, \underline{\underline{d}} : \underline{\underline{d}}$$

From reduced Clausius-Duhem inequality

$$\rho \dot{\psi} \leq 2\mu \underbrace{\underline{\underline{d}} : \underline{\underline{d}}}_{>0}$$

$\Rightarrow$ only if $\mu > 0$ energy is dissipated during the flow $\dot{\psi} < 0$

Kinetic Energy of Fluid Motion

Dissipation of kinetic energy in ideal and Newtonian fluids.

First some useful results:

1) Integration by parts in fixed domain Ω

with "no slip" boundaries $\underline{v} = \underline{0}$ on $\partial\Omega$.

$$\boxed{\int_\Omega (\nabla_x^2 \underline{v}) \cdot \underline{v} \, dV_x = -\int_\Omega (\nabla_x \underline{v}) : (\nabla_x \underline{v}) \, dV_x}$$

To see this consider $(v_{i,j} v_i)_{,j} = v_{i,jj} v_i + v_{i,j} v_{i,j}$

$$(\nabla_x^2 \underline{v}) \cdot v = v_{i,jj} v_i = (v_{i,j} v_i)_{,j} - v_{i,j} v_{i,j}$$
$$= \nabla \cdot ((\nabla_x \underline{v})^T \underline{v}) - (\nabla_x \underline{v}) : (\nabla_x \underline{v})$$

substituting into integral and applying div-thm

$$\int_\Omega (\nabla_x^2 \underline{v}) \cdot \underline{v} \, dV_x = \int_{\partial\Omega} (\nabla_x v)^T \cancelto{0}{\underline{v}} \cdot \underline{n} \, dA_x - \int_\Omega (\nabla_x \underline{v}) : (\nabla_x \underline{v}) \, dV_x$$

2) Poincaré Inequality

$$\| \underline{u} \|_\Omega \leq \lambda \, \| \nabla u \|_\Omega \quad \text{for} \quad \underline{u} = 0 \quad \partial\Omega \quad \lambda > 0$$

using standard inner product

$$\boxed{\int_\Omega |\underline{u}|^2 \, dV_x \leq \lambda \int_\Omega \nabla \underline{u} : \nabla \underline{u} \, dV_x}$$

Notice λ has units of L^2 and scales with area of Ω.

Kinetic Energy of Newtonian & Ideal fluids

Consider a fixed domain Ω with $\underline{v} = 0$ on $\partial\Omega$

and a conservative body force $b = -\nabla_x \Phi$.

The kinetic energy is given by

$$K(t) = \int_\Omega \frac{1}{2} \rho_0 |\underline{v}|^2 \, dV_x \quad \text{and} \quad K(0) = K_0$$

I) Newtonian fluid

$$\boxed{K(t) \leq e^{-2\mu t/\lambda \rho_0} K_0}$$

The kinetic energy of a Newtonian fluid dissipates to zero exponentially fast.

II) Ideal fluid

$$\boxed{K(t) = K_0}$$

The kinetic energy of ideal fluid is constant.

By def. of K we have

$$\frac{d}{dt}K(t) = \int_\Omega \frac{1}{2}\rho_0 \frac{d}{dt}|\underline{v}|^2 \, dV_x = \int_\Omega \rho_0 \, \dot{\underline{v}} \cdot \underline{v} \, dV_x$$

from Navier-Stokes Eqns: $\rho_0 \dot{\underline{v}} = \mu \nabla_x^2 \underline{v} - \nabla \Psi$

$$\frac{d}{dt}K(t) = \int_\Omega (\mu \nabla_x^2 \underline{v} - \nabla \Psi) \cdot \underline{v} \, dV_x$$

show $\int_\Omega \nabla_x \psi \cdot \underline{v} \, dV_x = 0$

$$\nabla_x \cdot (\varphi \underline{v}) = \nabla_x \varphi \cdot \underline{v} + \cancelto{0}{(\nabla_x \cdot \underline{v})} \varphi = \nabla_x \varphi \cdot \underline{v}$$

substitute and use Div-Thm

$$\frac{d}{dt}K(t) = \int_\Omega \mu (\nabla_x^2 \underline{v}) \cdot \underline{v} \, dV_x - \int_{\partial\Omega} \psi \cancelto{0}{\underline{v}} \cdot \underline{n} \, dA_x$$

using integration by parts

$$\frac{d}{dt}K(t) = -\mu \int_\Omega (\nabla_x \underline{v}) : (\nabla_x \underline{v}) \, dV_x$$

for Ideal fluid $\mu = 0 \Rightarrow K(t) = K_0$

for Newtonian fluid apply Pointcare inequality

$$\frac{d}{dt}K(t) \leq -\frac{\mu}{\lambda} \int_\Omega |\underline{v}|^2 dV_x = -\frac{2\mu}{\lambda \rho_0} K(t)$$

so that we have

$$\boxed{\frac{d}{dt}K(t) \leq -\frac{2\mu}{\lambda \rho_0} K(t)}$$

where λ depends on area of the domain.

Solve by seperation of parts

$$\frac{dK}{K} \leq -\frac{2\mu}{\rho_0 \lambda}\, dt = -\alpha\, dt$$

$$\ln K \leq -\alpha t + c_0$$

$$K \leq c_1 e^{-\alpha t}$$

Initial condition $\quad K(0) \leq c_1 = K_0$

$$\Rightarrow \quad K(t) \leq K_0 e^{-\frac{2\mu}{\lambda \rho_0} t} \quad ✓$$

In absence of fluid motion on the boundary fluid motion decays exponentially.

The rate of decay depends

$$\boxed{\nu = \frac{\mu}{\rho_0}} \quad \text{kinematic viscosity}$$

Scaling Navier Stokes Equations

$$\rho_0 \frac{\partial \underline{v}}{\partial t} + (\nabla_x \underline{v})\, \underline{v} = \mu \nabla_x^2 \underline{v} - \nabla_x p + \rho g$$

reduced pressure:

$$-\nabla_x p + \rho g = -\nabla_x p - \rho g \hat{z} = -\nabla(p + \rho g z) = -\nabla \pi$$

we have

$$\rho_0 \left(\frac{\partial \underline{v}}{\partial t} + (\nabla_x \underline{v})\, \underline{v} \right) - \mu \nabla_x^2 \underline{v} = -\nabla_x \pi$$

Non-dimensionalize with generic quantities to define standard dimensionless parameters.

- Dependent variables: $\underline{v}$, π
- Independent variables: $\underline{x}$, t
- Parametess: $\rho \left[\frac{M}{L^3}\right]$ $\mu \left[\frac{M}{LT}\right] \rightarrow \nu = \frac{\mu}{\rho} \left[\frac{L^2}{T}\right]$

  + Geometry, BC, IC

Use parameters to scale the variables:

$$\underline{v}' = \frac{\underline{v}}{v_c} \qquad \pi' = \frac{\pi}{\pi_c} \qquad \underline{x}' = \frac{\underline{x}}{x_c} \qquad t' = \frac{t}{t_c}$$

substitute into governing equations

$$\frac{\rho_o v_c}{t_c}\frac{\partial \underline{v}'}{\partial t'} + \frac{\rho v_c^2}{x_c}(\nabla_x' \underline{v}')\underline{v}' - \frac{\mu v_c}{x_c^2}\nabla_x'^2 \underline{v}' = -\frac{\pi_c}{x_c}\nabla_x' \pi'$$

Option 1: Scale to accumulation term

$$\frac{\partial \underline{v}'}{\partial t'} + \underbrace{\frac{v_c t_c}{x_c}}_{\Pi_1}(\nabla_x' \underline{v}')\underline{v}' - \underbrace{\frac{\nu t_c}{x_c^2}}_{\Pi_2}\nabla_x'^2 \underline{v}' = -\underbrace{\frac{\pi_c t_c}{x_c \rho_o v_c}}_{\Pi_3}\nabla_x' \pi'$$

where $\nu = \frac{\mu}{\rho}$ "momentum diffusivity"

Three dimensionless groups ⇒ define time scale

$\Pi_1 = \frac{v_c t_c}{x_c} = 1$ ⇒ advective scale $t_c = t_A = \frac{x_c}{v_c}$

$\Pi_2 = \frac{\nu t_c}{x_c^2} = 1$ ⇒ diffusive scale $t_c = t_D = \frac{x_c^2}{\nu}$

Use Π_3 to define pressure scale

$\Pi_3 = \frac{\pi_c t_c}{x_c \rho_o v_c} = 1$ ⇒ $\pi_c = \frac{x_c \rho_o v_c}{t_c}$

Choose a diffusive time scale $t_c = \frac{x_c^2}{\nu}$

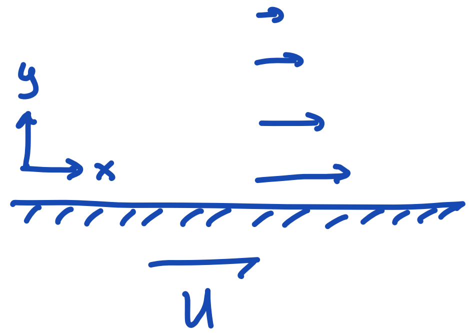
$$\frac{\partial \underline{v}}{\partial t} + \frac{v_c x_c}{\nu}(\nabla_x' \underline{v}')\underline{v}' - \nabla_x'^2 \underline{v} = -\nabla_x' \pi'$$

⇒ one remaining dim. less group

$\boxed{Pe_m = \frac{t_D}{t_A} = \frac{v_c x_c}{\nu} = Re}$ Reynolds number

Rayleigh's problem

- Semi-infinite half-space
- Stationary fluid
- Impulsively started plate with velocity U.



$$v_c = U \quad \rightarrow \quad Re = \frac{U x_c \rho_0}{\mu} \ll 1 \quad \Rightarrow \quad U \ll \frac{\mu}{x_c \rho_0}$$

But what is x_c? Not obvious

Redimensionalize assuming $Re \ll 1$

$$\frac{\partial \underline{v}}{\partial t} - \nu \nabla^2 \underline{v} = -\nabla \pi \quad \& \quad \nabla \cdot \underline{v} = 0 \qquad \underline{v} = \begin{pmatrix} u \\ w \end{pmatrix}$$

Simplify the equations:

Domain is infinite in x but $|\pi| < \infty \Rightarrow \frac{\partial \pi}{\partial x} = 0$

Flow is horizontal: $\underline{v} = \begin{pmatrix} u \\ w \end{pmatrix} \Rightarrow w = 0$

From continuity: $\frac{\partial u}{\partial x} + \cancelto{0}{\frac{\partial w}{\partial y}} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0 \Rightarrow u = u(y)$

$$\nabla^2 \underline{v} = v_{i,jj} \, \underline{e}_i \qquad i,j \in \{1,2\}$$

$$= \begin{pmatrix} v_{1,11} & v_{1,22} \\ v_{2,11} & v_{2,22} \end{pmatrix} = \begin{pmatrix} \cancelto{0}{u_{xx}} & u_{yy} \\ \cancelto{0}{w_{xx}} & \cancelto{0}{w_{yy}} \end{pmatrix} = \begin{pmatrix} u_{yy} \\ 0 \end{pmatrix}$$

Substituting we have

x-mom.: $$\frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial x^2} = 0$$

y-mom.: $$0 = -\frac{\partial \pi}{\partial y}$$

$$\Rightarrow \quad \boxed{\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}} \qquad \text{with} \quad u(0,y) = 0 \qquad u(t,0) = U$$

This is identical to heating a semi-infinite rod from the end.

Problem has self-similar solution in

$$\eta = \frac{y}{\sqrt{4\nu t}} \quad \text{and} \quad u(y,t) = U f(\eta)$$

where $\sqrt{4\nu t}$ takes role of char. length that depends on t.

derivatives: $$\frac{\partial \eta}{\partial t} = -\frac{1}{2}\frac{\eta}{t} \qquad \frac{\partial \eta}{\partial y} = \frac{1}{\sqrt{4\nu t}}$$

The derivatives of u transform as:

$$\frac{\partial u}{\partial t} = U \frac{df}{d\eta}\frac{\partial \eta}{\partial t} = -\frac{U}{2}\frac{\eta}{t} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = U \frac{d^2 f}{d\eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 = \frac{U}{4\nu t}\frac{d^2 f}{d\eta^2}$$

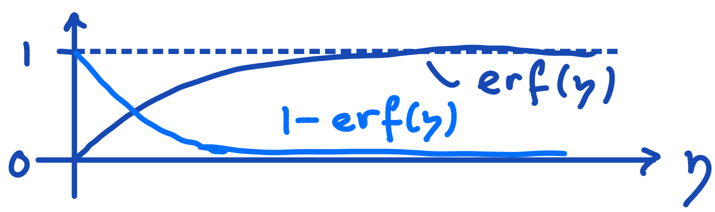
substituting into PDE:

$$\frac{d^2 f}{d\eta^2} + 2\eta \frac{df}{d\eta} = 0 \qquad \text{with} \quad f(\eta = 0) = 1$$

Reduce PDE in y and t to ODE in η

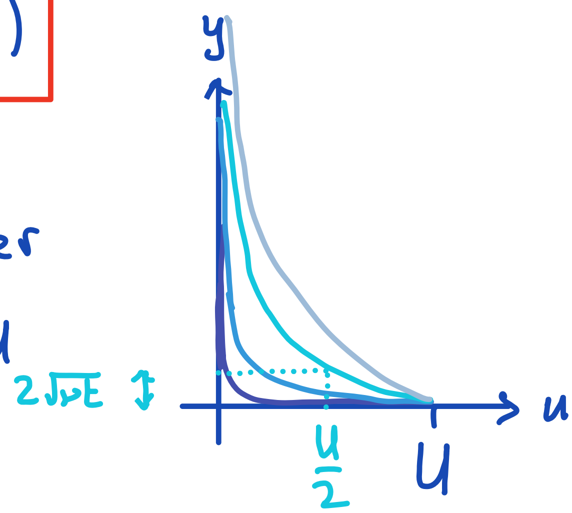
Solution : $f(\eta) = 1 - \mathrm{erf}(\eta)$ (Gauss)

where $\mathrm{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\xi^2} d\xi$ error function



Resubstituting for $f = \frac{u}{U}$ and $\eta = \frac{y}{\sqrt{4\nu t}}$

$$\boxed{u(y,t) = U\left(1 - \mathrm{erf}\left(\frac{y}{\sqrt{4Dt}}\right)\right)}$$



Diffusive boundary layer where momentum added by boundary penetrates into the quiescent fluid.

$\nu = \frac{\mu}{\rho_0}$ is Diffusion coefficient.