Orthogonal tensors

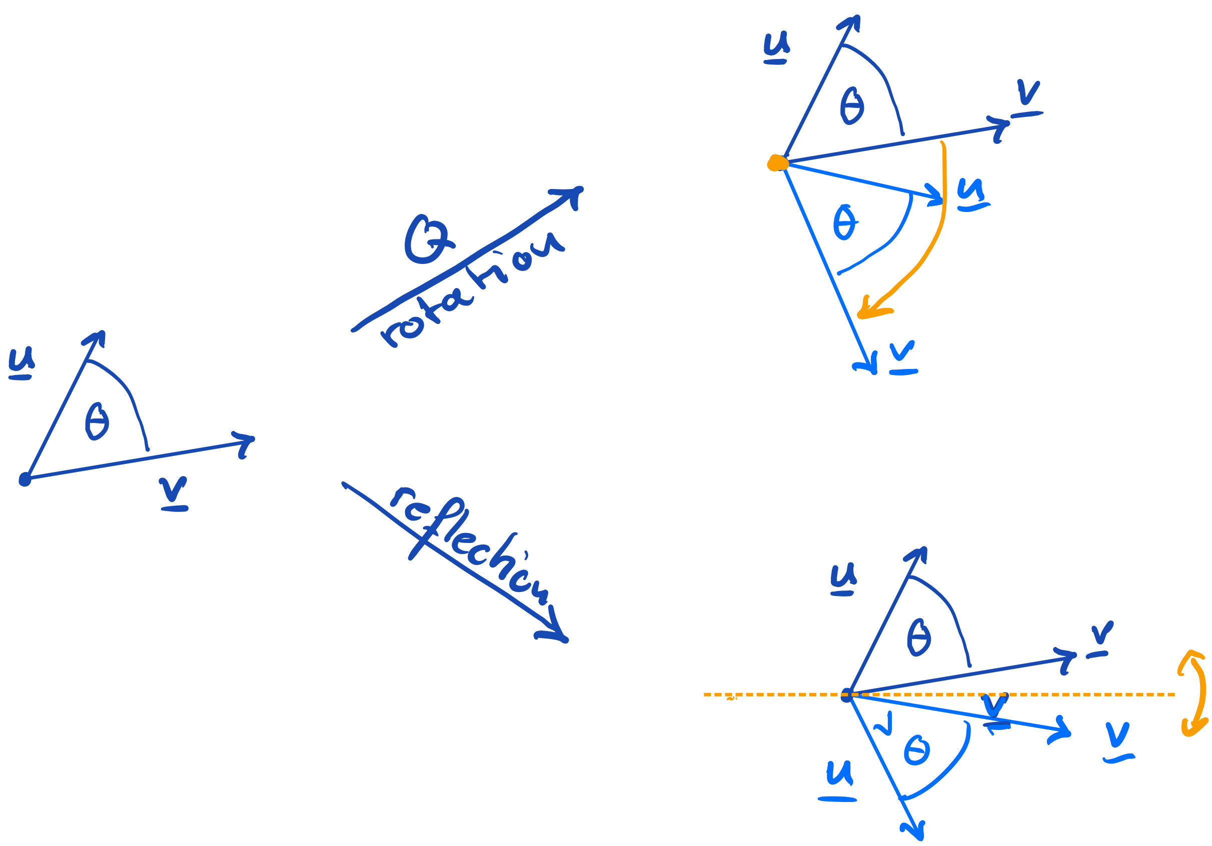
An orthogonal tensor $\underline{\underline{Q}} \in \mathcal{V}^2$ is a linear transformation satisfying

$$\boxed{\underline{\underline{Q}}\,\underline{u} \cdot \underline{\underline{Q}}\,\underline{v} = \underline{u} \cdot \underline{v}} \qquad \text{for all } \underline{u}, \underline{v} \in \mathcal{V}$$

$$\underline{u} \cdot \underline{v} = |\underline{u}|\,|\underline{v}| \cos\theta$$

$\Rightarrow$ preserves length & angle

only two possible operations:



Properties of orthogonal matrices:

$$\boxed{\begin{gathered} \underline{\underline{Q}}^T = \underline{\underline{Q}}^{-1} \\ \underline{\underline{Q}}^T \underline{\underline{Q}} = \underline{\underline{Q}}\,\underline{\underline{Q}}^T = \underline{\underline{I}} \\ \det(\underline{\underline{Q}}) = \pm 1 \end{gathered}}$$

Example: $1 = \det(\underline{\underline{I}}) = \det(\underline{\underline{Q}}^T \underline{\underline{Q}})$

$$= \det(\underline{\underline{Q}}^T)\det(\underline{\underline{Q}}) = \det(\underline{\underline{Q}})^2$$

$$\Rightarrow \quad \det(\underline{\underline{Q}}) = \pm 1$$

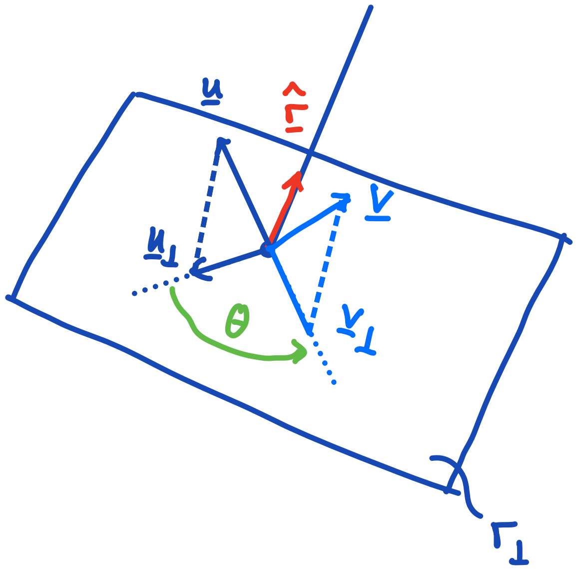
If $\det(\underline{\underline{Q}}) = 1 \Rightarrow$ rotation

$\det(\underline{\underline{Q}}) = -1 \Rightarrow$ reflection

In mechanics we are mostly concerned with rotations.

Rotation Matrices

$$\underline{v} = Q(\hat{\underline{r}}, \theta)\, \underline{u}$$



$\hat{\underline{r}}$ = axis of rotation

$r_\perp$ = plane $\perp$ to $\hat{\underline{r}}$

θ = counter clock wise angle

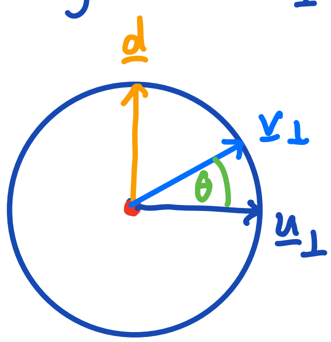
$$\underline{u} = \underline{u}_\parallel + \underline{u}_\perp$$

$$\underline{v} = \underline{v}_\parallel + \underline{v}_\perp$$

$$\underline{v}_\parallel = \underline{u}_\parallel = (\underline{u} \cdot \hat{\underline{r}})\hat{\underline{r}} = (\hat{\underline{r}} \otimes \hat{\underline{r}})\, \underline{u}$$

What is $\underline{v}_\perp$?

looking onto $r_\perp$



$$\underline{d} = \hat{\underline{r}} \times \underline{u}$$

$\underline{d} \perp \underline{u}_\perp \Rightarrow$ basis in $r_\perp$

$$\Rightarrow \underline{v}_\perp = \cos\theta\, \underline{u}_\perp + \sin\theta\, \underline{d}$$

Rotated vector:

$$\underline{v} = \underline{v}_\parallel + \underline{v}_\perp = (\hat{\underline{r}} \otimes \hat{\underline{r}})\underline{u} + \cos\theta\, (\underline{\underline{I}} - \hat{\underline{r}} \otimes \hat{\underline{r}})\underline{u} + \sin\theta\, \hat{\underline{r}} \times \underline{u}$$

Can we write: $\underline{v} = \underline{\underline{Q}}(\hat{\underline{r}}, \theta)\, \underline{u}$?

Axial Tensor

Need to write $\underline{r} \times \underline{u} = \underline{\underline{R}}\,\underline{u}$!

$\underline{\underline{R}}\,\underline{u} = R_{ij} u_j \underline{e}_i$ and $\underline{r} \times \underline{u} = \epsilon_{mnl} r_m u_n \underline{e}_l$

$R_{ij} u_j \underline{e}_i = \epsilon_{mnl} r_m u_n \underline{e}_l$!

all indices are dummies $\Rightarrow$ rename

$l \to i$: $R_{ij} u_j \underline{e}_i = \epsilon_{mni} r_m u_n \underline{e}_i$

$R_{ij} u_j = \epsilon_{mni} r_m u_n$ $\quad i$ = free index

$n \to j$: $R_{ij} u_j = \epsilon_{mji} r_m u_j$

$R_{ij} = \epsilon_{mji} r_m$ $\quad i, j$ = free $\quad m$ = dummy

$m \to k$: $R_{ij} = \epsilon_{kji} r_k$

prop. of ϵ: $\epsilon_{kji} = -\epsilon_{jki} = \epsilon_{ikj}$

$$\boxed{R_{ij} = \epsilon_{ikj} r_k}$$

$tr(\underline{\underline{R}}) = 0$

$$\underline{\underline{R}} = R_{ij} \underline{e}_i \otimes \underline{e}_j = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix}$$

$\underline{\underline{R}} = -\underline{\underline{R}}^T$ skew sym.

$R_{12} = \epsilon_{132} r_3 = -r_3 \quad R_{13} = \epsilon_{123} r_2 = r_2 \quad R_{23} = \epsilon_{213} = -r_1$

Back to rotation

$$\underline{v} = (\underline{r}\otimes\underline{r})\underline{u} + \cos\theta\,(\underline{\underline{I}} - \underline{r}\otimes\underline{r})\underline{u} + \sin\theta\,\underline{\underline{R}}\,\underline{u}$$

$$= \underbrace{\left[\underline{r}\otimes\underline{r} + \cos\theta\,(\underline{\underline{I}} - \underline{r}\otimes\underline{r}) + \sin\theta\,\underline{\underline{R}}\right]}_{\underline{\underline{Q}}(\underline{r},\theta)}\,\underline{u}$$

Euler representation of finite rotation tensor

$$\boxed{\begin{aligned}\underline{\underline{Q}}(\underline{r},\theta) &= \underline{r}\otimes\underline{r} + \cos\theta\,(\underline{\underline{I}} - \underline{r}\otimes\underline{r}) + \sin\theta\,\underline{\underline{R}}\\ Q_{ij}(\underline{r},\theta) &= r_i r_j + \cos\theta\,(\delta_{ij} - r_i r_j) + \sin\theta\,\epsilon_{ikj} r_k\end{aligned}}$$

Example: Rotation tensor around $\underline{e}_3$

$$\underline{\underline{Q}}(\underline{e}_3,\theta) = \underline{e}_3\otimes\underline{e}_3 + \cos\theta\,(\underline{\underline{I}} - \underline{e}_3\otimes\underline{e}_3) + \sin\theta\,\underline{\underline{R}}_3$$

$$= \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix} + \cos\theta\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix} + \sin\theta\begin{bmatrix} 0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\underline{\underline{Q}}(\underline{e}_3,\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

Rotate $\underline{e}_1$ by 90° $\left(\frac{\pi}{2}\right)$ counter clockwise

$$\cos\left(\frac{\pi}{2}\right) = 0 \qquad \sin\left(\frac{\pi}{2}\right) = 1$$

$$\underline{\underline{Q}}\left(\underline{e}_3, \frac{\pi}{2}\right) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\underline{Q}}\left(\underline{e}_3, \frac{\pi}{2}\right) \underline{e}_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \underline{e}_2 \checkmark$$

Determine θ and $\underline{r}$ from $\underline{\underline{Q}}$:

Rotation angle θ:

$$tr(\underline{\underline{Q}}) = Q_{ii} = \underbrace{e_{3,i}\, e_{3,i}} + \cos\theta\,(\underbrace{\delta_{ii}} - \underbrace{e_{3,i}\, e_{3,i}}) + \sin\theta\, \cancelto{0}{\epsilon_{iki}}\, r_k$$

$$= 1 + \cos\theta\,(3 - 1)$$

$$\Rightarrow \qquad \boxed{\cos\theta = \frac{tr(\underline{\underline{Q}}) - 1}{2}}$$

Example: $\underline{\underline{Q}}\left(\underline{e}_3, \frac{\pi}{2}\right) \qquad tr(\underline{\underline{Q}}) = 1$

$$\cos\theta = 0 \quad \Rightarrow \quad \theta = \frac{\pi}{2}$$

Axis of rotation $\underline{r}$:

$$\underline{\underline{Q}} = \mathrm{sym}(\underline{\underline{Q}}) + \mathrm{skew}(\underline{\underline{Q}})$$

$$\mathrm{sym}(\underline{\underline{Q}}) = \tfrac{1}{2}(\underline{\underline{Q}} + \underline{\underline{Q}}^T) = \underline{r} \otimes \underline{r} + \cos\theta\,(\underline{\underline{I}} - \underline{r} \otimes \underline{r})$$

$$\mathrm{skew}(\underline{\underline{Q}}) = \tfrac{1}{2}(\underline{\underline{Q}} - \underline{\underline{Q}}^T) = \sin\theta\ \underline{\underline{R}} = \sin\theta\ \epsilon_{ikj}\, r_k\ \underline{e}_i \otimes \underline{e}_j$$

$$\underline{\underline{R}} = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix}$$ is axial tensor

but we are given $\underline{\underline{Q}}$ not $\underline{\underline{R}}$

$$\mathrm{skew}(\underline{\underline{Q}}) = \tfrac{1}{2}(Q_{ij} - Q_{ji})\ \underline{e}_i \otimes \underline{e}_j$$

$$\mathrm{skew}(\underline{\underline{Q}}) = \underbrace{\sin\theta\ \epsilon_{ikj}\, r_k}_{[\mathrm{skew}(\underline{\underline{Q}})]_{ij}}\ \underline{e}_i \otimes \underline{e}_j$$

equate two expressions for components

$$\underbrace{\tfrac{1}{2}(Q_{ij} - Q_{ji})}_{\text{know}} = \sin\theta\ \epsilon_{ikj}\ \underset{\uparrow\ \text{want}}{r_k}$$

remove ϵ_{ikj} using $\epsilon\delta$ identities

$$\epsilon_{ilj} \; \frac{1}{2}\left(Q_{ij} - Q_{ji}\right) = \sin\theta \; \epsilon_{ilj} \; \epsilon_{ikj} \, r_k$$

$$= \sin\theta \; \epsilon_{lij} \, \epsilon_{kij} \, r_k$$

$$= \sin\theta \; 2 \, \delta_{lk} \, r_k$$

$$= \sin\theta \, 2 \, r_l$$

$$\Rightarrow \quad \boxed{\begin{aligned} r_l &= \frac{\epsilon_{ilj}\left(Q_{ij} - Q_{ji}\right)}{4 \sin\theta} \\[1em] \underline{r} &= \frac{1}{2\sin\theta} \begin{bmatrix} Q_{32} - Q_{23} \\ Q_{13} - Q_{31} \\ Q_{21} - Q_{12} \end{bmatrix} \end{aligned}}$$

Example: $\underline{\underline{Q}}\left(\underline{e}_3, \frac{\pi}{2}\right) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\underline{r} = \frac{1}{2 \cancelto{1}{\sin\left(\frac{\pi}{2}\right)}} \begin{bmatrix} 0 - 0 \\ 0 - 0 \\ 1 + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \underline{e}_3$$

Infinitesimal Rotations

$$\lim_{\theta \to 0} \underline{\underline{Q}}(\underline{\hat{r}}, \theta) = (\hat{r} \otimes \hat{r}) + \cancelto{1}{\cos\theta} \, (\underline{\underline{I}} - \underline{\hat{r}} \otimes \underline{\hat{r}}) + \cancelto{\theta}{\sin\theta} \, \underline{\underline{R}}$$

$$= \underline{\underline{I}} + \theta \, \underline{\underline{R}}$$

$\Rightarrow$ Axial tensor $\underline{\underline{R}}$ give infinitesimal rotation

$$\underline{v} = (\underline{\underline{I}} + \theta \, \underline{\underline{R}}) \, \underline{u}$$

$$\underline{v} = \underline{u} + \theta (\underline{\hat{r}} \times \underline{u})$$

$\Rightarrow$ cross product gives infinitesimal rotation