Differentiation of Tensor fields

A field is a function of space.

scalar fields: $\phi(\underline{x})$ temp., density

vector fields: $\underline{v}(\underline{x})$ force, velocity

tensor fields: $\underline{\underline{S}}(\underline{x})$ stress, conductivity

Today's lecture is review and extension of concepts from multivariable calculus.

Gradients

Gradient of scalar field

Scalar field $\phi(\underline{x})$ is differentiable at $\underline{x}$ if there exists a vector field $\nabla\phi \in \mathcal{V}$ such that

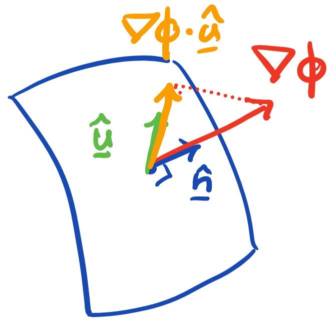
$$\boxed{\phi(\underline{x}+\underline{h}) = \phi(\underline{x}) + \nabla\phi(\underline{x})\cdot\underline{h} + \mathcal{O}(|h|)}$$

by Taylor expansion. Or equivalently

$$\boxed{\nabla\phi(\underline{x})\cdot\hat{\underline{u}} = \frac{d}{d\varepsilon}\,\phi(\underline{x}+\epsilon\hat{\underline{u}})\Big|_{\epsilon=0}} \quad \text{for all } \underline{v}\in\mathcal{V}$$

where $\underline{h} = \epsilon\,\hat{\underline{u}}$ and $|\hat{\underline{u}}| = 1$.

The vector $\nabla\phi$ is called the gradient of ϕ.



Consider a level set of ϕ

$\nabla\phi \parallel \hat{\underline{n}}$ in direction of increasing ϕ

$$\hat{\underline{n}} = \frac{\nabla\phi}{|\nabla\phi|}$$

Directional derivative (Gâteaux operator)

$$\boxed{D_{\hat{\underline{u}}}\phi(\underline{x}) = \frac{d}{d\epsilon}\,\phi(\underline{x} + \epsilon\,\hat{\underline{u}})\Big|_{\epsilon=0} = \nabla\phi(\underline{x})\cdot\hat{\underline{u}}}$$

Representation of the gradient in frame $\{\underline{e}_i\}$

$$\phi(\bar{\underline{x}} + \epsilon\,\underline{u}) = \phi(\underbrace{\bar{x}_1 + \epsilon u_1}_{x_1}, \underbrace{\bar{x}_2 + \epsilon u_2}_{x_2}, \underbrace{\bar{x}_3 + \epsilon u_3}_{x_3})$$

$$\begin{aligned}
\nabla\phi\cdot\hat{\underline{u}} &= \frac{d}{d\epsilon}\,\phi(\bar{x}_1 + \epsilon u_1, \bar{x}_2 + \epsilon u_2, \bar{x}_3 + \epsilon u_3)\Big|_{\epsilon=0} \\
&= \frac{d\phi}{dx_1}\frac{dx_1}{d\epsilon} + \frac{d\phi}{dx_2}\frac{dx_2}{d\epsilon} + \frac{d\phi}{dx_3}\frac{dx_3}{d\epsilon}\Big|_{\epsilon=0} \\
&= \frac{d\phi}{dx_1}u_1 + \frac{d\phi}{dx_2}u_2 + \frac{d\phi}{dx_3}u_3 \\
&= \frac{\partial\phi}{\partial x_i}u_i = \phi_{,i}\,u_i = \phi_{,i}\,u_j\,\delta_{ij} = \phi_{,i}\,u_j\,\underline{e}_i\cdot\underline{e}_j
\end{aligned}$$

$$\nabla\phi\cdot\hat{\underline{u}} = (\phi_{,i}\,\underline{e}_i)\cdot(u_j\,\underline{e}_j) \quad \checkmark$$

Note: Index notation for derivatives

$$\boxed{\frac{\partial \phi}{\partial x_i} = \phi_{,i}}$$ derivative index after comma!

Gradient in components: $[\nabla \phi] = \phi_{,i}\, \underline{e}_i = \begin{pmatrix} \frac{\partial \phi}{\partial x_1} \\ \frac{\partial \phi}{\partial x_2} \\ \frac{\partial \phi}{\partial x_3} \end{pmatrix}$

Gradient of a vector field

A vector field $\underline{v}(x) \in \mathcal{V}$ is differentiable at $\underline{x}$ if there exists a tensor field $\nabla \underline{v}(\underline{x}) \in \mathcal{V}^2$ such that

$$\underline{v}(\underline{x}+\underline{h}) = \underline{v}(\underline{x}) + \nabla \underline{v}(x)\, \underline{h} + o(|h|)$$

by Taylor expansion or equivalently

$$\boxed{\nabla \underline{v}\, \hat{\underline{u}} = \frac{d}{d\epsilon} \underline{v}(\underline{x} + \epsilon \hat{\underline{u}})\Big|_{\epsilon=0}} \quad \text{for all } \underline{u} \in \mathcal{V}$$

where $\underline{h} = \epsilon \hat{\underline{u}}$

In frame $\{\underline{e}_i\}$ we write components of $\underline{v}$ as $v_i = v_i(x_1, x_2, x_3)$. For any scalar ϵ and unit vector $\hat{\underline{u}} = u_k \underline{e}_k$ at $\bar{\underline{x}} = \bar{x}_j \underline{e}_j$

we have the i-th component

$$v_i(\underline{\bar{x}} + \epsilon \underline{\hat{u}}) = v_i(\bar{x}_1 + \epsilon u_1, \bar{x}_2 + \epsilon u_2, \bar{x} + \epsilon u_3)$$

by the chain rule

$$\frac{d}{d\epsilon} v_i(\underline{\bar{x}} + \epsilon \underline{\hat{u}}) = \frac{\partial v_i}{\partial x_1} u_1 + \frac{\partial v_i}{\partial x_2} u_2 + \frac{\partial v_i}{\partial x_3} u_3 = \frac{\partial v_i}{\partial x_j} u_j$$

For full vector $\underline{v} = v_i \underline{e}_i$

$$\nabla \underline{v}\, \underline{\hat{u}} = \frac{d}{d\varepsilon} \underline{v}(\underline{\bar{x}} + \epsilon \underline{\hat{u}})\Big|_{\epsilon=0} = \frac{d}{d\varepsilon}\left(v_i(\underline{\bar{x}} + \epsilon \underline{\hat{u}}) \underline{e}_i\right)\Big|_{\epsilon=0}$$

$$= \frac{d}{d\varepsilon}\left(v_i(\underline{\bar{x}} + \epsilon \underline{\hat{u}})\right)\Big|_{\epsilon=0} \underline{e}_i = \frac{\partial v_i}{\partial x_j} u_j \underline{e}_i$$

components: $\boxed{[\nabla \underline{v}]_{ij} = \frac{\partial v_i}{\partial x_j} = v_{i,j}}$

Representation $\nabla \underline{v} = v_{i,j}\, \underline{e}_i \otimes \underline{e}_j$

Explicit

$$\nabla \underline{v} = \begin{bmatrix} v_{1,1} & v_{1,2} & v_{1,3} \\ v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,1} & v_{3,2} & v_{3,3} \end{bmatrix} = \begin{bmatrix} \nabla v_1^T \\ \nabla v_2^T \\ \nabla v_3^T \end{bmatrix}$$

Divergence of a vector field

Def: To any $\underline{v}(x) \in \mathcal{V}$ we associate a scalar field $\nabla \cdot \underline{v}$ called the divergence of $\underline{v}$

$$\boxed{\nabla \cdot \underline{v} = \mathrm{tr}(\nabla \underline{v})}$$

In frame $\{\underline{e}_i\}$ with $\underline{v}(\underline{x}) = v_i(\underline{x})\,\underline{e}_i$ we have

$$\boxed{\nabla \cdot \underline{v} = \mathrm{tr}(\nabla \underline{v}) = v_{i,i}}$$

If $\nabla \cdot \underline{v} = 0$ a field is solenoidal or divergence free. If $\underline{v}$ is a displacement or velocity then $\nabla \cdot \underline{v}$ is related to (rate of) volume change.

Divergence of a tensor field

To any $\underline{\underline{S}}(\underline{x}) \in \mathcal{V}^2$ we associate a vector field $\nabla \cdot \underline{\underline{S}} \in \mathcal{V}$ called the divergence of $\underline{\underline{S}}$

$$\boxed{(\nabla \cdot \underline{\underline{S}}) \cdot \underline{a} = \nabla \cdot (\underline{\underline{S}}^T \underline{a})} \quad \text{for all } \underline{a} \in \mathcal{V}$$

uses definition of vector divergence!

In frame $\{\underline{e}_i\}$ with $\underline{\underline{S}} = S_{ij}\, \underline{e}_i \otimes \underline{e}_j$ and $\underline{a} = a_k \underline{e}_k$

we have $\underline{q} = \underline{\underline{S}}^T \underline{a}$ or $q_j = S_{ij}\, a_i$ $\quad (q_i = S_{ji}\, a_j)$

substituting

$$(\nabla\cdot\underline{\underline{S}})\underline{a} = \nabla\cdot(\underline{\underline{S}}^T\underline{a}) = \nabla\cdot\underline{q} = tr(\nabla\underline{q}) = q_{j,j}$$

$$= S_{ij,j}\, a_i = (S_{ij,j}\, \underline{e}_i)\cdot(a_k \underline{e}_k)$$

by the arbitraryness of $\underline{a}$ we have

$$\boxed{\nabla\cdot\underline{\underline{S}} = S_{ij,j}\, \underline{e}_i}$$

Gradient & Divergence product rules

$\phi \in \mathbb{R}$, $\quad \underline{v} \in \mathcal{V}$, $\quad \underline{\underline{S}} \in \mathcal{V}^2$

$$\boxed{\begin{aligned}
\nabla\cdot(\phi\,\underline{v}) &= \underline{v}\cdot\nabla\phi + \phi\,\nabla\cdot\underline{v} \\
\nabla\cdot(\phi\,\underline{\underline{S}}) &= \underline{\underline{S}}\,\nabla\phi + \phi\,\nabla\cdot\underline{\underline{S}} \\
\nabla\cdot(\underline{\underline{S}}^T\underline{v}) &= (\nabla\cdot\underline{\underline{S}})\cdot\underline{v} + \underline{\underline{S}} : \nabla\underline{v} \\
\nabla(\phi\,\underline{v}) &= \underline{v}\otimes\nabla\phi + \phi\,\nabla\underline{v}
\end{aligned}}$$

Note: Last identity is gradient.

Example: $\nabla \cdot (\underline{\underline{S}}^T \underline{v})$ note $\underline{\underline{S}} = \underline{\underline{S}}(x)$ and $\underline{v} = v(\underline{x})$

$$q(x) = \underline{\underline{S}}^T(\underline{x})\, \underline{v}(\underline{x}) \qquad q_j = S_{ij} v_i$$

$$\nabla \cdot \underline{q} = tr(\underline{q}) = q_{j,j} = (S_{ij} v_i)_{,j}$$

$$= S_{ij,j} v_i + S_{ij} v_{i,j}$$

$$= (\nabla \cdot \underline{\underline{S}}) \cdot \underline{v} + \underline{\underline{S}} : \nabla \underline{v} \quad \checkmark$$

$\rightarrow$ useful for energy balance!

Example: $\nabla(\phi \underline{v}) = (\phi v_i)_{,j}\, \underline{e}_i \otimes \underline{e}_j$

$$= (\phi_{,j} v_i + \phi v_{i,j})\, \underline{e}_i \otimes \underline{e}_j$$

$$= v_i \phi_{,j}\, \underline{e}_i \otimes \underline{e}_j + \phi\, v_{i,j}\, \underline{e}_i \otimes \underline{e}_j$$

$$= \underline{v} \otimes \nabla \phi + \phi \nabla \underline{v} \quad \checkmark$$

Curl of a vector field

To any $\underline{v}(\underline{x}) \in \mathcal{V}$ we associate another vector field $\nabla \times \underline{v}$ defined by

$$\boxed{(\nabla \times \underline{v}) \times \underline{a} = (\nabla \underline{v} - \nabla \underline{v}^T)\, \underline{a}} \quad \text{for all } \underline{a} \in \mathcal{V}$$

Here $\underline{w} = \nabla \times \underline{v}$ is the axial vector of

$\underline{\underline{T}} = \nabla \underline{v} - \nabla \underline{v}^T = 2\,\mathrm{skew}(\nabla \underline{v})$

In index notation

$$w_j = \tfrac{1}{2} \epsilon_{ijk} T_{ik} = \tfrac{1}{2} \epsilon_{ijk} (v_{i,k} - v_{k,i})$$

$$= \tfrac{1}{2} (\epsilon_{ijk} v_{i,k} - \epsilon_{ijk} v_{k,i}) \qquad \epsilon_{ijk} = -\epsilon_{kji}$$

$$= \tfrac{1}{2} (\epsilon_{ijk} v_{i,k} + \epsilon_{kji} v_{k,i}) \qquad \text{flip } i \Leftrightarrow k \text{ in second}$$

$$w_j = \epsilon_{ijk} v_{i,k}$$

$$\Rightarrow \qquad \boxed{\underline{w} = \nabla \times \underline{v} = \epsilon_{ijk} v_{i,k} \underline{e}_j}$$

Note: Equivalently $\nabla \times \underline{v} = -\epsilon_{ijk} v_{i,j} \underline{e}_k$

by switching & renaming indices

Explicitly: $\nabla \times \underline{v} = (v_{3,2} - v_{2,3})\underline{e}_1 + (v_{1,3} - v_{3,1})\underline{e}_2$
$+ (v_{2,1} - v_{1,2})\underline{e}_3$

Physical interpretation:

If $\underline{v}$ is a velocity field then $\nabla \times \underline{v}$ measures the angular velocity.

If $\nabla \times \underline{v} = 0 \Rightarrow \underline{v}(x)$ is irrotational/conservative

Further we can show

$\boxed{\nabla \times \nabla \phi = \underline{0}}$ and $\boxed{\nabla \cdot (\nabla \times \underline{v}) = 0}$ $\Rightarrow$ HW3

this follows as

$$\nabla \times \nabla \phi = \nabla \times (\phi_{,i}\, \underline{e}_i) = \epsilon_{ijk} (\phi_{,i})_{,k}\, \underline{e}_j$$

$$= \epsilon_{ijk}\, \phi_{,ik}\, \underline{e}_j$$

$$= \tfrac{1}{2} (\epsilon_{ijk}\, \phi_{,ik} + \epsilon_{ijk}\, \phi_{,ik})\, \underline{e}_j$$

2^{nd} term $\epsilon_{ijk} = -\epsilon_{kji}$

$$= \tfrac{1}{2} (\epsilon_{ijk}\, \phi_{,ik} - \epsilon_{kji}\, \phi_{,ik})\, \underline{e}_j$$

$\phi_{,ik} = \phi_{,ki}$

$$= \tfrac{1}{2} (\epsilon_{ijk}\, \phi_{,ik} - \epsilon_{kji}\, \phi_{,ki})\, \underline{e}_j$$

rename dummys in second term $i \leftrightarrow j$

$$= \tfrac{1}{2} (\epsilon_{ijk}\, \phi_{,ik} - \epsilon_{ijk}\, \phi_{,ik})\, \underline{e}_j$$

$$= \underline{0}$$

Laplacian

To any scalar field $\phi \in \mathbb{R}$ we associate another scalar field $\Delta\phi = \nabla^2\phi$ defined by

$$\boxed{\Delta\phi = \nabla^2\phi = \nabla\cdot\nabla\phi}$$

In frame $\{\underline{e}_i\}$ with $\nabla\phi = \phi_{,i}\,\underline{e}_i$ we have

$$\nabla\cdot\nabla\phi = tr(\nabla\nabla\phi) = tr(\phi_{,ij}\,\underline{e}_i\otimes\underline{e}_j) = \phi_{,ii}$$

$$\boxed{\nabla^2\phi = \phi_{,ii}}$$

Scalar Laplacian governs steady heat flow.

To any vector field $\underline{v}(x) \in \mathcal{V}$ we associate another vector field $\Delta\underline{v} = \nabla^2\underline{v} \in \mathcal{V}$ defined by $\boxed{\Delta\underline{v} = \nabla^2\underline{v} = \nabla\cdot\nabla\underline{v}}$

In frame $\{\underline{e}_i\}$ with $\underline{v} = v_i \underline{e}_i$, $\nabla \underline{v} = v_{i,j}\, \underline{e}_i \otimes \underline{e}_j$

and $\nabla \cdot \underline{\underline{S}} = S_{ij,j}\, \underline{e}_i$ we have

$$\boxed{\Delta \underline{v} = v_{i,jj}\, \underline{e}_i}$$

Vector Laplacian governs viscous flow.

There are several useful identities. One commonly used relation

$$\nabla^2 \underline{v} = \nabla(\nabla \cdot \underline{v}) - \nabla \times (\nabla \times \underline{v})$$

if $\underline{v}(x)$ is both solenoidal $(\nabla \cdot \underline{v} = 0)$ and irrotational $(\nabla \times \underline{v} = 0)$ then $\nabla^2 \underline{v} = 0$ and $\underline{v}$ is harmonic.

Used in derivation of incompressible Navier-Stokes eqn.

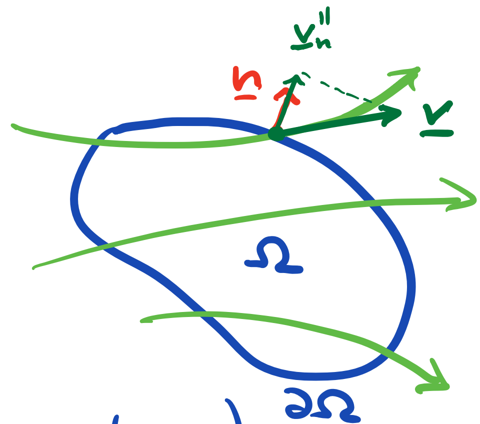
Integral theorems

Essential to derive balance laws

Vector divergence theorem

For any $\underline{v}(\underline{x}) \in \mathcal{V}$ we have

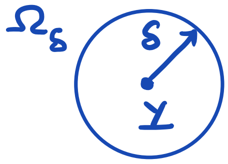
$$\boxed{\begin{aligned} \int_{\partial\Omega} \underline{v} \cdot \underline{n} \, dA &= \int_{\Omega} \nabla \cdot \underline{v} \, dV \\ \int_{\partial\Omega} v_i n_i \, dA &= \int_{\Omega} v_{i,i} \, dV \end{aligned}}$$



(for proof see vector calculus class)

Physical Interpretation:

Here $\underline{v}$ is either a velocity $[\frac{L}{T}]$ or a volumetric flux $[\frac{L^3}{L^2 T} = \frac{L}{T}]$. The units of $\int_{\partial\Omega} \underline{v} \cdot \underline{n} \, d\Omega$ are then $[\frac{L^3}{T}]$ so that the l.h.s. represents the rate at which volume is leaving or entering Ω.



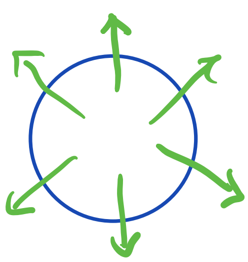
$$\int_{\partial\Omega_\delta} \underline{v}\cdot\underline{n}\, dA = \int_{\Omega_\delta} \nabla\cdot\underline{v}\, dV$$

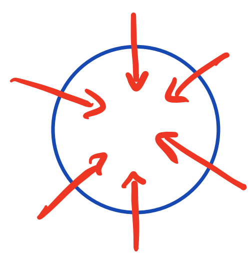
$$\lim_{\delta\to 0} \int_{\Omega_\delta} \nabla\cdot\underline{v}\, dV = V_\delta \, \nabla\cdot\underline{v}\big|_{\underline{x}} \qquad V_\delta = \text{vol. of sphere}$$

$$\boxed{\nabla\cdot\underline{v}\big|_{\underline{x}} = \lim_{\delta\to 0} \frac{1}{V_\delta} \int_{\partial\Omega} \underline{v}\cdot\underline{n}\, dA}$$

Divergence is the point wise rate of volume expansion/contraction.

$\nabla\cdot\underline{v} > 0$





$\nabla\cdot\underline{v} < 0$

Incompressible flows/deformations are solenoidal $\nabla\cdot\underline{v} = 0$.

Tensor divergence theorem

For any $\underline{\underline{S}}(\underline{x}) \in \mathcal{V}^2$ on domain Ω with boundary $\partial\Omega$ we have

$$\boxed{\begin{aligned} \int_{\partial\Omega} \underline{\underline{S}}\,\underline{n}\, dA &= \int_{\Omega} \nabla \cdot \underline{\underline{S}}\, dV \\ \int_{\partial\Omega} S_{ij} n_j \, dA &= \int_{\Omega} S_{ij,j}\, dV \end{aligned}}$$

To derive this from vector divergence Thm consider arbitrary constant vector $\underline{a} \in \mathcal{V}$

$$\underline{a} \cdot \int_{\partial\Omega} \underline{\underline{S}}\,\underline{n}\, dA = \int_{\partial\Omega} \underline{a} \cdot \underline{\underline{S}}\, n \, dA = \int_{\partial\Omega} (\underline{\underline{S}}^T \underline{a}) \cdot \underline{n}\, dA$$

where $\underline{\underline{S}}^T \underline{a}$ is a vector and we can apply vector divergence Thm

$$\int_{\partial\Omega} (\underline{\underline{S}}^T \underline{a}) \cdot \hat{\underline{n}}\, dA = \int_{\Omega} \nabla \cdot (\underline{\underline{S}}^T \underline{a})\, dV$$

using the definition: $(\nabla \cdot \underline{\underline{S}}) \cdot \underline{a} = \nabla \cdot (\underline{\underline{S}}^T \underline{a})$

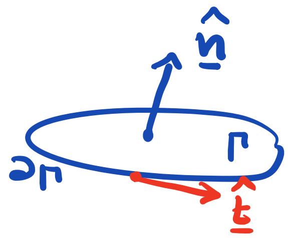
$$\int_{\partial\Omega} (\underline{\underline{S}}^T \underline{a}) \cdot \hat{\underline{n}}\, dA = \int_{\Omega} (\nabla \cdot \underline{\underline{S}}) \cdot \underline{a}\, dV$$

using def. of transpose and that $\underline{a}$ is const.

$$\underline{a} \cdot \int_{\partial\Omega} \underline{\underline{S}}\,\hat{\underline{n}}\, dA = \underline{a} \cdot \int_{\Omega} \nabla \cdot \underline{\underline{S}}\, dV$$

the result follows from arbitrariness of $\underline{a}$

Stokes Thm



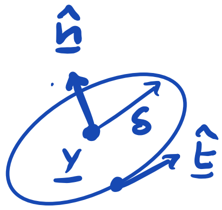
Consider surface Γ with boundary $\partial\Gamma$, unit normal $\hat{\underline{n}}$ and unit tangent (right-handed).

Then for any $\underline{v}(\underline{x}) \in \mathcal{V}$ we have

$$\boxed{\int_{\Gamma} (\nabla \times \underline{v}) \cdot \hat{\underline{n}}\, dA = \oint_{\partial\Gamma} \underline{v} \cdot \hat{\underline{t}}\, ds}$$

Here $\oint_{\partial\Gamma} \underline{v} \cdot \hat{\underline{t}}\, ds$ is the circulation of $\underline{v}$ around $\partial\Gamma$.

Physical Interpretation:



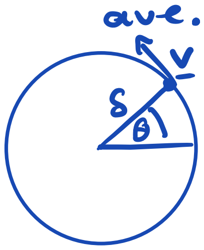
Γ_δ is a disk of radius δ around $\underline{y}$.

$$\oint_{\partial\Gamma} \underline{v}(\underline{x}) \cdot \hat{\underline{t}}(\underline{x})\, ds = \int_{\Gamma} (\nabla \times \underline{v})(\underline{x}) \cdot \hat{\underline{n}}\, dA$$

In the limit of $\delta \to 0$

$$\underbrace{\overline{\underline{v} \cdot \hat{\underline{t}}}\Big|_{\underline{y}}}_{\text{ave. tangential velocity}} 2\pi\delta \approx \nabla \times \underline{v}\Big|_{\underline{y}} \cdot \hat{\underline{n}}\ \pi\delta^2$$

ave. tangential velocity ~ angular velocity



angular velocity: $\omega = \frac{d\theta}{dt}$

$|\underline{v}| = \omega\delta$

$$\Rightarrow \quad \overline{\underline{v} \cdot \hat{\underline{t}}}\Big|_{\underline{y}} = \omega\delta$$

$$2\pi\delta^2\omega \approx \nabla \times \underline{v}\Big|_{\underline{y}} \cdot \hat{\underline{n}}\ \pi\delta^2$$

$$2\omega = \nabla \times \underline{v}\Big|_{\underline{y}} \cdot \hat{\underline{n}} \qquad \hat{\underline{n}} = \frac{\nabla \times \underline{v}}{|\nabla \times \underline{v}|}\Bigg|_{\underline{y}}$$

$$2\omega = \frac{(\nabla \times \underline{v}|_{\underline{y}}) \cdot (\nabla \times \underline{v}|_{\underline{y}})}{|\nabla \times \underline{v}|_{\underline{y}}} = |\nabla \times \underline{v}|_{\underline{y}}$$

$$\Rightarrow \quad \boxed{|\nabla \times \underline{v}|_{\underline{y}} = 2\omega}$$

Curl of $\underline{v}$ is twice the angular velocity.