Temperature-dependent viscosity

Common source of non-linearity is the variation of viscosity with temperature.

Ice rheology is complex and depends on the microscopic deformation mechanism.

We consider "diffusion creep" which results in a Newtonian rheology.

$$\boxed{\mu = \frac{RTd^2}{42 V_m D_{0,V}} \exp\left(\frac{E_A}{RT}\right)}$$

Parameters: d = grain diameter $\sim 1\,mm$

T = temperature

V_m = molar volume $1.97 \cdot 10^{-5}\ \frac{m^3}{mol}$

$D_{0,V}$ = vol. diff. constant $9.1 \cdot 10^{-4}\ \frac{m^2}{s}$

E_A = vol. diff. act. energy $59.4\ \frac{kJ}{mol}$

R = mol. gas constant $8.314\ \frac{J}{K\,mol}$

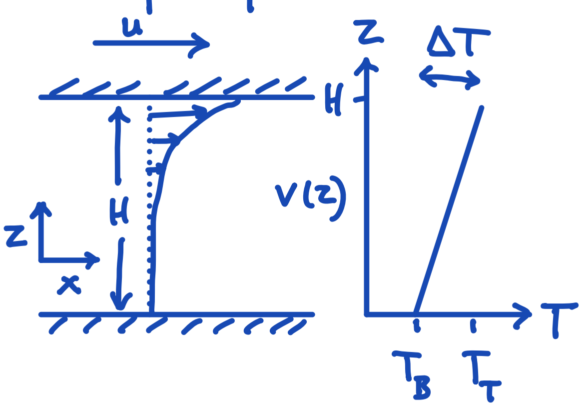
Newtonian because $\mu \neq \mu(\underline{v})$!

But μ has Arrhenius dependence on T

The temperature-dependence of the pre-exponential factor is often neglected.

$$\boxed{\mu = \mu_0 \exp\left(\frac{E_A}{RT}\right)} \qquad \mu_0 = \frac{R T_m d^2}{42\, V_m D_{0,v}}$$

Example problem: Couette flow with T gradient



Boudary layer forms at the hot top boundary where shear is localized.

In the absence of heating by viscous dissipation the T-field is independent of velocity

$\Rightarrow$ one-way coupling: $v = v(T)$ but $T \neq T(v)$

Viscous energy dissipation leads to two-way coupl.

Energy balance equation

$$\rho c_p \frac{\partial T}{\partial t} + \nabla \cdot [\underline{v}\, \rho c_p T - \kappa \nabla T] = \underline{\underline{\sigma}} : \underline{\underline{d}} + \rho r$$

Assumptions:

1) Neglect heat production: r.h.s $= 0$

2) Steady state $\frac{\partial T}{\partial t} = 0$

3) Physical parameters are constant (ρ, c_p, κ)

$$\Rightarrow \quad \nabla \cdot [\underline{v} T - \alpha \nabla T] = 0 \qquad \alpha = \frac{\kappa}{\rho c_p} \text{ therm. diffusivity}$$

4) Flow is incompressible

5) $\underline{v} = \begin{bmatrix} v(z) \\ 0 \end{bmatrix}$ and $T = T(z)$ $\Rightarrow$ $\nabla T = \begin{bmatrix} 0 \\ \frac{\partial T}{\partial z} \end{bmatrix}$

$$\nabla \cdot [\underline{v} T] = \underline{v} \cdot \nabla T + T(\underbrace{\nabla \cdot \underline{v}}_{0})$$

$$= [v(z) \quad 0] \begin{bmatrix} 0 \\ \frac{\partial T}{\partial z} \end{bmatrix} = \underline{0}$$

$$\Rightarrow \quad -\nabla \cdot \alpha \nabla T = 0 \quad \Rightarrow \quad \boxed{\frac{d^2 T}{dz^2} = 0}$$

Boundary conditions: $T(0) = T_B \qquad T(H) = T_T \qquad T_T \geq T_0$

Temperature field: $\Delta T = T_T - T_B$

Integrating twice: $T = T_B + \frac{\Delta T}{H} z$

Velocity & pressure fields:

$$-\nabla \cdot [\mu (\nabla \underline{v} + \nabla^T \underline{v})] + \nabla \pi = 0 \qquad \pi = p + \rho g z$$

$$\nabla \cdot \underline{v} = 0$$

deviatoric stress in component form:

$$\underline{\underline{\tau}} = \mu (\nabla \underline{v} + \nabla^T \underline{v}) = \mu \begin{bmatrix} 2 v_{x,x} & v_{x,z} + v_{z,x} \\ v_{x,z} + v_{z,x} & 2 v_{z,z} \end{bmatrix}$$

From flow geometry: $v_z = 0 \rightarrow v_{z,z} = v_{z,x} = 0$

$v_{x,x} = 0$ (from continuity)

$$\nabla \pi = \begin{pmatrix} \pi_{,x} \\ \pi_{,z} \end{pmatrix} = \begin{pmatrix} \pi_{,x} \\ 0 \end{pmatrix}$$

$\Rightarrow$ all terms in z-momentum balance vanish

$$\boxed{-\frac{\partial}{\partial z}\left[\mu \frac{\partial v}{\partial z}\right] + \frac{\partial \pi}{\partial x} = 0} \qquad v = v_x$$

In Couette example the domain is infinite in x-dir, so that $\frac{\partial \pi}{\partial x} = 0$.

Solve following ODE:

$$\boxed{\begin{array}{l} \frac{\partial}{\partial z}\left[\mu(T(z))\frac{\partial v}{\partial z}\right] = 0 \\ v(0) = 0 \qquad v(H) = u \\ \mu = \mu_0 \exp\left(\frac{E_a}{RT}\right) \\ T = T_B + \frac{\Delta T}{H} z \\ \Delta T = T_T - T_B > 0 \end{array}}$$

where (the last three lines in the box above)

Integrate once:

$\mu \frac{\partial v}{\partial z} = c_1$ here $c_1 = \tau$ = shear stress

$\tau = \mu \frac{\partial v}{\partial z}$ is definition of μ !

Integrate once more:

$$v(z) = \tau \int_0^z \frac{dz}{\mu(T(z))}$$

$$v(z) = \tau \int_0^z \frac{dz}{\mu_0 \exp\left(\frac{E_a}{RT(z)}\right)} = \frac{\tau}{\mu_0} \int_0^z \exp\left(\frac{-E_a/R}{T(z)}\right) dz$$

$$v(z) = \frac{\tau}{\mu_0} \int_0^z \exp\left(\frac{-E_a/R}{T_B + \Delta T z/H}\right) dz$$

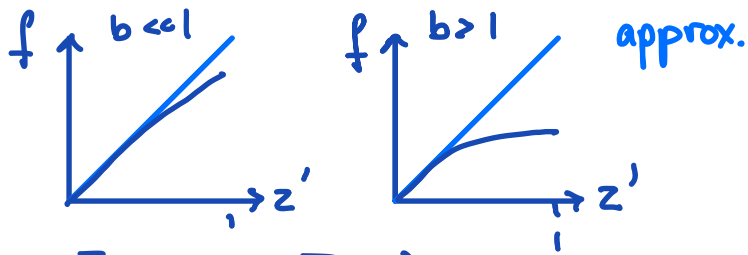
difficult integral, but if $\Delta T \ll T_B$ the exponential factor can be approximated as

$$f(z) = -\frac{E_a}{RT_B}\,\frac{1}{1+\frac{\Delta T}{T_B}\frac{z}{H}} = \frac{-a}{1+bz'} \qquad z' = \frac{z}{H} \quad a = \frac{E_a}{RT_B} \quad b = \frac{\Delta T}{T_B}$$

Taylor series expansion at $z'=0$

$$L_0 f(z) = f(0) + \left.\frac{df}{dz'}\right|_0 z' = -a + abz = -a(1-bz')$$

where $\frac{df}{dz} = \frac{ab}{(1+bz')^2}$



so that we have

$$f(z) = -\frac{E_a}{RT_B}\,\frac{1}{1+\frac{\Delta T}{T_B}\frac{z}{H}} \approx -\frac{E_a}{RT_B}\left(1-\frac{\Delta T}{T_B}\frac{z}{H}\right)$$

$$V(z) = \frac{\tau}{\mu_0}e^{-a}\int_0^z e^{abz'}\,dz \qquad dz' = \frac{dz}{H} \quad dz = H\,dz'$$

$$V(z') = \frac{\tau H}{\mu_0}e^{-a}\int_0^{z'} e^{abz''}\,dz'' = \frac{\tau H}{\mu_0}\frac{e^{-a}}{ab}\left(e^{abz'} - e^{0}\right)$$

$$V(z') = \frac{\tau H}{\mu_0}\frac{e^{-a}}{ab}\left(e^{abz'} - 1\right)$$

1) Set velocity of top plate and find shear stress τ

2) Set shear stress τ and find velocity of top plate

$$V(z'=1) = u = \frac{\tau H}{\mu_0}\frac{e^{-a}}{ab}\left(e^{ab}-1\right)$$

$$\Rightarrow \quad \tau = \frac{u\mu_0}{H}\frac{ab}{e^{-a}}\frac{1}{e^{ab}-1}$$

substitute

$$\frac{V(z')}{u} = \frac{e^{abz'}-1}{e^{ab}-1} \qquad \text{where } a = \frac{E_a}{RT_B} \quad b = \frac{\Delta T}{T_B} \quad z' = \frac{z}{H}$$

so that $\quad a \cdot b = \dfrac{E_a \Delta T}{R T_B^2}$

Hence the velocity profile is:

$$\boxed{\frac{V(z')}{u} = \frac{\exp\left(\frac{E\Delta T}{RT_B^2}\frac{z}{H}\right)-1}{\exp\left(\frac{E\Delta T}{RT_B^2}\right)-1}}$$

$$T = T_B + \frac{\Delta T}{H}z = T_B\left(1+\frac{\Delta T}{T_B}\frac{z}{H}\right) = T_B(1-$$

$$\frac{T}{T_B} = 1 - bz'$$

corresponding viscosity

$$\mu = \mu_0 \exp\left(\frac{E_a}{RT}\right)$$

$$\frac{\mu}{\mu_0} = \exp\left(\frac{E_a}{RT_B}\frac{T_B}{T}\right) = \exp\left(\frac{E_a}{RT_B}\frac{1}{1-bz'}\right) = \exp\left(\frac{a}{1-bz'}\right)$$