Cauchy-Green Strain Tensor

For $\underline{x} = \underline{\varphi}(\underline{X})$ with $\underline{\underline{F}} = \nabla\varphi$,

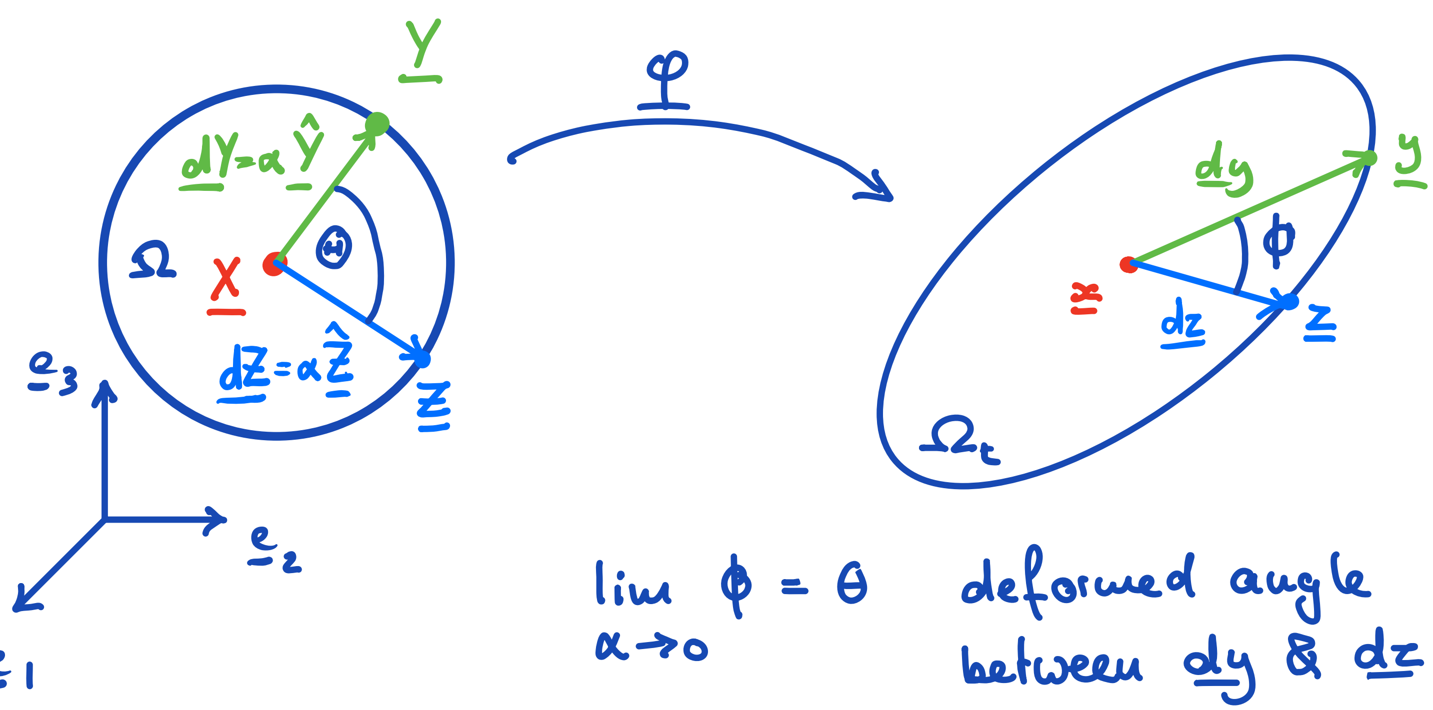
$$\boxed{\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}} = \underline{\underline{U}}^2}$$ sym. pos. def.

$\underline{\underline{U}}$ is right-strech tensor $\quad \underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}}$

$\Rightarrow$ only information about streches

Interpretation of $\underline{\underline{C}}$

How are changes in relative position and orientation of material points quantified by $\underline{\underline{C}}$?



$\lim\limits_{\alpha\to 0} \phi = \Theta$ deformed angle between dy & dz

Cauchy-Green strain relations

For any point $\underline{X} \in B$ and unit vectors $\hat{\underline{Y}}$ and $\hat{\underline{Z}}$ we define $\lambda(\hat{\underline{Y}}) > 0$ and $\theta(\hat{\underline{Y}}, \hat{\underline{Z}}) \in [0, \pi]$ by

$$\boxed{\lambda(\hat{\underline{Y}}) = \sqrt{\hat{\underline{Y}} \cdot \underline{\underline{C}}\,\hat{\underline{Y}}}} \quad \text{and} \quad \boxed{\cos\theta(\hat{\underline{Y}}, \hat{\underline{Z}}) = \frac{\hat{\underline{Y}} \cdot \underline{\underline{C}}\,\hat{\underline{Z}}}{\sqrt{\hat{\underline{Y}} \cdot \underline{\underline{C}}\,\hat{\underline{Y}}}\,\sqrt{\hat{\underline{Z}} \cdot \underline{\underline{C}}\,\hat{\underline{Z}}}}}$$

I. Streches

1D: $\lambda = \frac{\ell}{L}$ ratio of deformed to initial length

3D: $\lambda(\hat{\underline{Y}}) = \frac{|d\underline{y}|}{|d\underline{Y}|}$ strech in direction $\hat{\underline{Y}}$ at $\underline{X}$.

To determine the stretch we use $d\underline{y} = \underline{\underline{F}}(\underline{X})\,d\underline{Y}$.

$$|d\underline{y}|^2 = d\underline{y} \cdot d\underline{y} = \underline{\underline{F}}\,d\underline{Y} \cdot (\underline{\underline{F}}\,d\underline{Y}) = d\underline{Y} \cdot \underline{\underline{F}}^T\underline{\underline{F}}\,d\underline{Y} = d\underline{Y} \cdot \underline{\underline{C}}\,d\underline{Y}$$
$$= \alpha^2\, \hat{\underline{Y}} \cdot \underline{\underline{C}}\, \hat{\underline{Y}}$$

$|d\underline{Y}|^2 = \alpha^2$ by definition

$$\Rightarrow \lambda^2(\hat{\underline{Y}}) = |d\underline{y}|^2 / |d\underline{Y}|^2 = \hat{\underline{Y}} \cdot \underline{\underline{C}}\,\hat{\underline{Y}}$$

so that $\lambda(\hat{\underline{Y}}) = \sqrt{\hat{\underline{Y}} \cdot \underline{\underline{C}}\,\hat{\underline{Y}}}$ ✓

If $\underline{u}_p$ is a right-principal strech, so that

$$\left(\underline{\underline{C}} - \lambda_p^2 \underline{\underline{I}}\right)\underline{u}_p = \underline{0} \quad \text{(no sum)}$$

$$\underline{u}_p \cdot \left(\underline{\underline{C}} - \lambda_p^2 \underline{\underline{I}}\right)\underline{u}_p = 0$$

$$\underline{u}_p \cdot \underline{\underline{C}}\,\underline{u}_p - \lambda_p^2 \cancelto{1}{\underline{u}_p \cdot \underline{u}_p} = 0$$

$$\underline{u}_p \cdot \underline{\underline{C}}\,\underline{u}_p = \lambda_p^2$$

$$\Rightarrow \quad \boxed{\lambda_p = \sqrt{\underline{u}_p \cdot \underline{\underline{C}}\,\underline{u}_p} \equiv \lambda(\underline{u}_p)} \qquad |\underline{u}_p| = 1$$

justifies referring to λ_i's as principal streches.

Arguments similar to determination of principal stresses show that $\lambda(\hat{\underline{Y}})$ has extremum if $\hat{\underline{Y}} = \hat{\underline{u}}_i$.

II. Shear

Change in angle

$$\boxed{\gamma(\hat{\underline{Y}}, \hat{\underline{Z}}) = \Theta(\hat{\underline{Y}}, \hat{\underline{Z}}) - \theta(\hat{\underline{y}}, \hat{\underline{z}})}$$

$\Theta(d\hat{\underline{Y}}, d\hat{\underline{Z}})$ angle between $d\hat{\underline{Y}}$ & $d\hat{\underline{Z}}$ in initial conf.

$\theta(d\hat{\underline{Y}}, d\hat{\underline{Z}})$ angle between $\underline{dy}$ & $\underline{dz}$ in limit $\alpha \to 0$

$$\boxed{\cos\phi \longrightarrow \cos\theta(\hat{\underline{Y}}, \hat{\underline{Z}})}$$

To see this consider $\underline{dy} \cdot \underline{dz} = |\underline{dy}||\underline{dz}| \cos\phi$

$$\Rightarrow \cos\phi = \frac{\underline{dy} \cdot \underline{dz}}{|\underline{dy}||\underline{dz}|}$$

where $\underline{dy} \cdot \underline{dz} = (\underline{\underline{F}}\, \underline{dY}) \cdot (\underline{\underline{F}}\, \underline{dZ})$

$$= \underline{dY} \cdot \underline{\underline{F}}^T \underline{\underline{F}}\, \underline{dZ} = \underline{dY} \cdot \underline{\underline{C}}\, \underline{dZ}$$

$$= \alpha^2\, \hat{\underline{Y}} \cdot \underline{\underline{C}}\, \hat{\underline{Z}} \quad \text{where } \underline{dY} = \alpha \hat{\underline{Y}} \text{ and } \underline{dZ} = \alpha \hat{\underline{Z}}$$

$$|\underline{dy}| = \sqrt{\underline{dy} \cdot \underline{dy}} = \alpha \sqrt{\hat{\underline{Y}} \cdot \underline{\underline{C}}\, \underline{Y}}$$

$$|\underline{dz}| = \sqrt{\underline{dz} \cdot \underline{dz}} = \alpha \sqrt{\hat{\underline{Z}} \cdot \underline{\underline{C}}\, \hat{\underline{Z}}}$$

substituting into $\cos\phi = \frac{\underline{dy} \cdot \underline{dz}}{|\underline{dy}||\underline{dz}|}$

$$\cos\phi = \frac{d\hat{\underline{Y}} \cdot \underline{\underline{C}}\, d\hat{\underline{Z}}}{\sqrt{d\hat{\underline{Y}} \cdot \underline{\underline{C}}\, d\hat{\underline{Y}}} \sqrt{d\hat{\underline{Z}} \cdot \underline{\underline{C}}\, d\hat{\underline{Z}}}} \xrightarrow[\alpha \to 0]{} \cos\theta(d\hat{\underline{Y}}, d\hat{\underline{Z}})$$

Compute the shear $\gamma(\hat{\underline{Y}}, \hat{\underline{Z}}) = \Theta(\hat{\underline{Y}}, \hat{\underline{Z}}) - \theta(\hat{\underline{Y}}, \hat{\underline{Z}})$

$\Rightarrow$ interpret components of $\underline{\underline{C}}$

Components of $\mathbf{C}$

$$\mathbf{C} = C_{IJ}\, \mathbf{e}_I \otimes \mathbf{e}_J \quad \Rightarrow \quad C_{IJ} = \mathbf{e}_I \cdot \mathbf{C}\, \mathbf{e}_J$$

I) Diagonal components:

$$C_{II} = \mathbf{e}_I \cdot \mathbf{C}\, \mathbf{e}_I \qquad (\text{no sum})$$

1st Cauchy-Green: $\lambda(\mathbf{Y}) = \sqrt{\mathbf{Y} \cdot \mathbf{C}\, \mathbf{Y}}$

$$\Rightarrow \quad C_{II} = \lambda^2(\mathbf{e}_I) \quad \checkmark$$

II) Off-diagonal components

$$C_{IJ} = \mathbf{e}_I \cdot \mathbf{C}\, \mathbf{e}_J$$

substitute into 2nd Cauchy-Green

$$\cos\theta(\mathbf{e}_I, \mathbf{e}_J) = \frac{\mathbf{e}_I \cdot \mathbf{C}\, \mathbf{e}_J}{\sqrt{\mathbf{e}_I \cdot \mathbf{C}\, \mathbf{e}_I}\, \sqrt{\mathbf{e}_J \cdot \mathbf{C}\, \mathbf{e}_J}} = \frac{C_{IJ}}{\lambda(\mathbf{e}_I)\, \lambda(\mathbf{e}_J)}$$

$$\Rightarrow \quad C_{IJ} = \lambda(\mathbf{e}_I)\, \lambda(\mathbf{e}_J) \cos\theta$$

shear: $\gamma(\underline{e}_I, \underline{e}_J) = \underbrace{\Theta(\underline{e}_I, \underline{e}_J)}_{} - \theta(\underline{e}_I, \underline{e}_J)$

$$= \frac{\pi}{2} - \theta(\underline{e}_I, \underline{e}_J)$$

$$\Rightarrow \theta(\underline{e}_I, \underline{e}_J) = \frac{\pi}{2} - \gamma(\underline{e}_I, \underline{e}_J)$$

substitute into C_{IJ}

$$C_{IJ} = \lambda(\underline{e}_I)\,\lambda(\underline{e}_J)\cos\left(\frac{\pi}{2} - \gamma(\underline{e}_I, \underline{e}_J)\right)$$
$$= \lambda(\underline{e}_I)\,\lambda(\underline{e}_J)\sin(\gamma(\underline{e}_I, \underline{e}_J))$$

Interpretation of components of $\underline{\underline{C}}$:

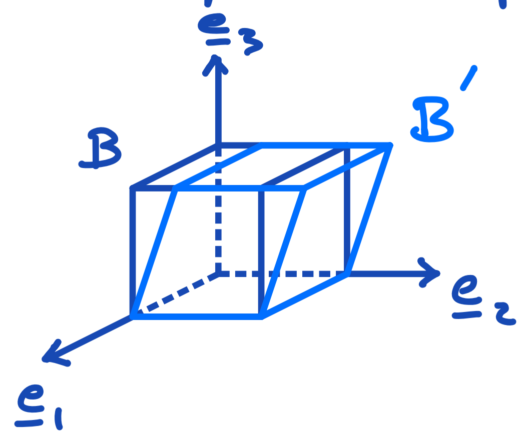
$$\boxed{\begin{aligned} C_{II} &= \lambda^2(\underline{e}_I) \\ C_{IJ} &= \lambda(\underline{e}_I)\,\lambda(\underline{e}_J)\sin\gamma(\underline{e}_I, \underline{e}_J) \end{aligned}}$$ (no sum)

diagonal $\rightarrow$ square of stretches in coord. dir.

off diagonal $\rightarrow$ shear between coord. dir.

The components of $\underline{\underline{C}}$ directly quantify stretch and shear unlike the components of $\underline{\underline{F}}$.

Example: Simple shear



$$\underline{x} = \underline{\varphi}(\underline{X}) = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 + \alpha X_3 \\ X_3 \end{bmatrix} \quad \alpha > 0$$

"simple shear in $\underline{e}_2$-$\underline{e}_3$ plane"

Deformation gradient:

$$\underline{\underline{F}} = \nabla \underline{\varphi} = \begin{pmatrix} \varphi_{1,1} & \varphi_{1,2} & \varphi_{1,3} \\ \varphi_{2,1} & \varphi_{2,2} & \varphi_{2,3} \\ \varphi_{3,1} & \varphi_{3,2} & \varphi_{3,3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow$ homogeneous deformation

Cauchy-Green strain tensor:

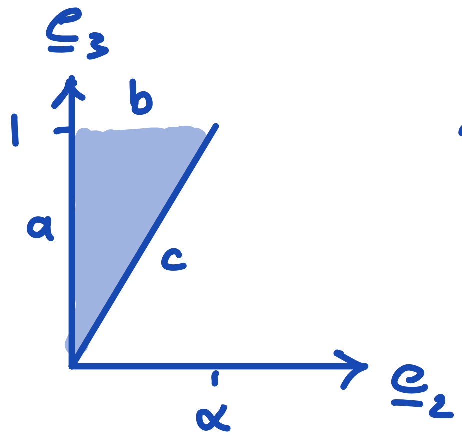
$$\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \alpha & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & \alpha & 1+\alpha^2 \end{bmatrix}}}$$

Stretches:

$C_{11} = \lambda^2(\underline{e}_1) = 1$ no strech

$C_{22} = \lambda^2(\underline{e}_2) = 1$ no strech

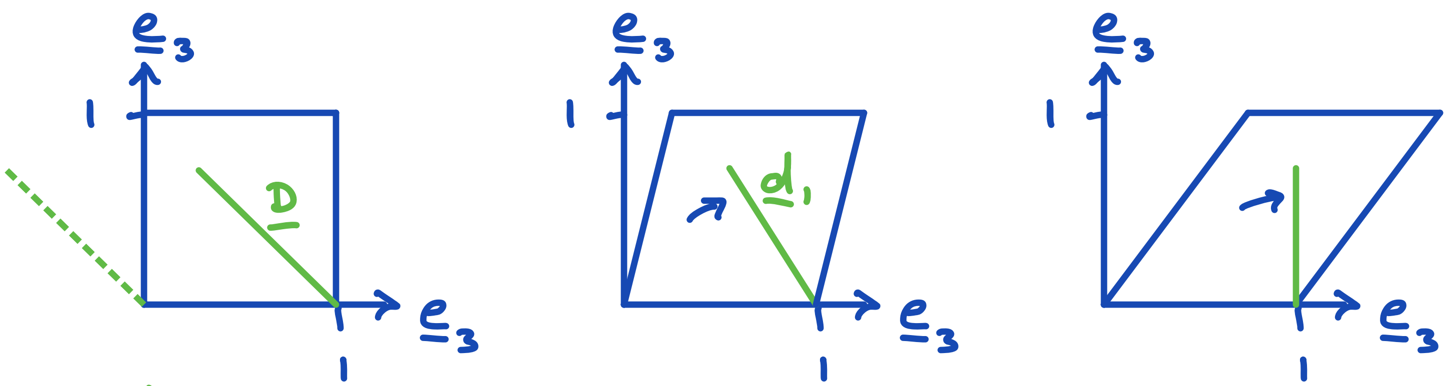
$C_{33} = \lambda^2(\underline{e}_3) = 1 + \alpha^2 \Rightarrow \lambda(\underline{e}_3) = \sqrt{1+\alpha^2}$



Pythagoras: $c = \sqrt{a^2 + b^2}$

$\lambda(\underline{e}_1) = \frac{c}{a} = c = \sqrt{1+\alpha^2}$

What about non-coordinate directions?



$\underline{D} = \frac{1}{\sqrt{2}}(\underline{e}_3 - \underline{e}_2)$

$\lambda(\underline{D})$? $\Rightarrow$ 1st Cauchy-Green SR

$$\lambda^2(\underline{D}) = \underline{D} \cdot \underline{\underline{C}}\,\underline{D}$$

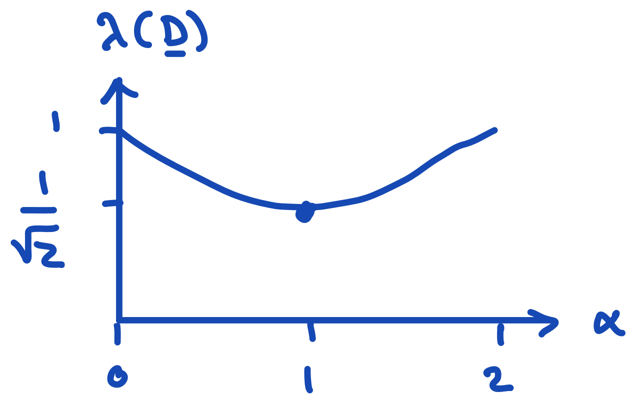
$$= \frac{1}{2}(\underline{e}_3 - \underline{e}_2) \cdot \underline{\underline{C}}(\underline{e}_3 - \underline{e}_2)$$

$$= \frac{1}{2}\left(\underline{e}_3 \cdot \underline{\underline{C}}\,\underline{e}_3 - \underline{e}_2 \cdot \underline{\underline{C}}\,\underline{e}_3 - \underline{e}_3 \cdot \underline{\underline{C}}\,\underline{e}_2 + \underline{e}_2 \cdot \underline{\underline{C}}\,\underline{e}_2\right)$$

$$= \frac{1}{2}\left(C_{33} - 2C_{23} + C_{22}\right) \qquad C_{23} = C_{32}$$

$$= \frac{1}{2}\left(1 + \alpha^2 - 2\alpha + 1\right)$$

$$\lambda^2(\underline{D}) = \frac{1}{2}\alpha^2 - \alpha + 1$$

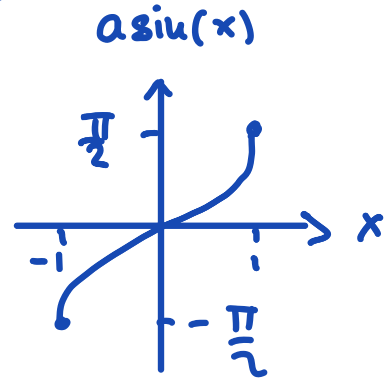


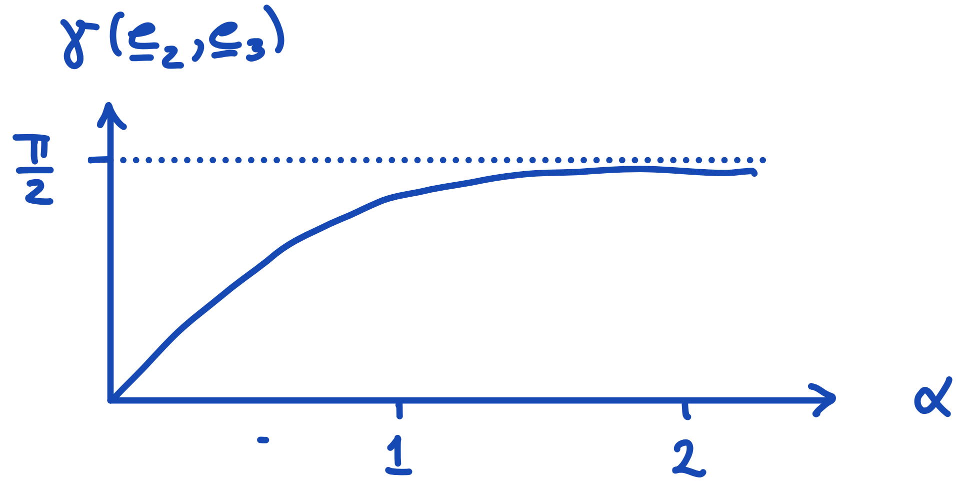
$\underline{\text{Shear}}$: $\quad C_{ij} \sim \sin(\gamma(\underline{e}_i, \underline{e}_j))$

$\Rightarrow \; C_{12} = C_{13} = 0 \quad$ no shear in these planes

$$C_{23} = \alpha = \cancelto{1}{\lambda(\underline{e}_2)}\,\cancelto{\sqrt{1+\alpha^2}}{\lambda(\underline{e}_3)}\,\sin(\gamma(\underline{e}_2, \underline{e}_3))$$

$$\gamma(\underline{e}_2, \underline{e}_3) = \operatorname{asin}\left(\frac{\alpha}{\sqrt{1+\alpha^2}}\right)$$





$$\lim_{\alpha \to \infty} \frac{\alpha}{\sqrt{1+\alpha}} = 1$$

Principal streches

$$(\underline{\underline{C}} - \mu_p \underline{\underline{I}})\, \underline{v}_p = \underline{0} \qquad \mu_p = \lambda_p^2$$

$$p(\mu) = \begin{vmatrix} 1-\mu & 0 & 0 \\ 0 & 1-\mu & \alpha \\ 0 & \alpha & 1+\alpha^2-\mu \end{vmatrix} = (1-\mu)^2(1+\alpha^2-\mu) - \alpha^2(1-\mu) = 0$$

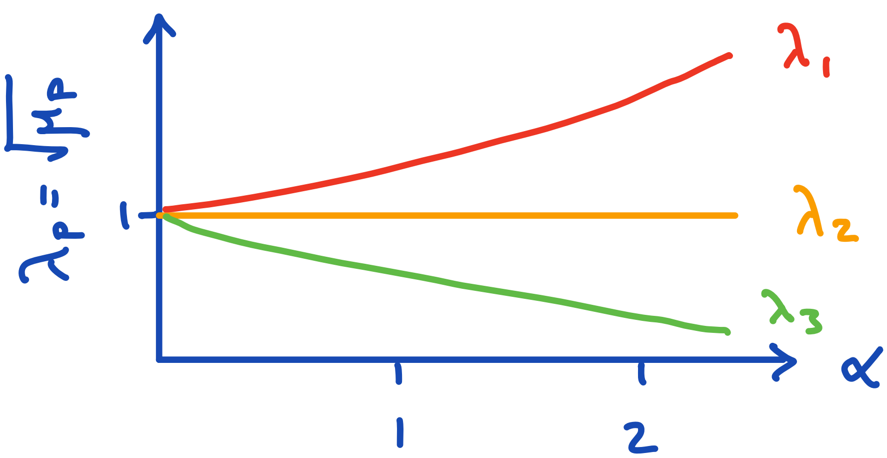
$$p(\mu) = (1-\mu)\left[(1-\mu)(1+\alpha^2-\mu) - \alpha^2\right] = 0$$

$$\Rightarrow \quad \mu_2 = 1$$

$$(1-\mu)(1+\alpha^2-\mu) - \alpha^2 = 0$$

$$\mu^2 - (1+\alpha^2)\mu + 1 = 0$$

$$\Rightarrow \quad \mu_{1/3} = 1 + \frac{1}{2}\alpha^2 \pm \frac{\alpha}{2}\sqrt{\alpha^2+4}$$



Principal directions:

$$(\underline{\underline{C}} - \mu_p \underline{\underline{I}})\, \underline{v}_p = \underline{0}$$

$\Rightarrow$

$$\underline{v}_1 = \begin{bmatrix} 0 \\ 1 \\ \frac{1}{2}(\alpha + \sqrt{1+\alpha^2}) \end{bmatrix}, \quad \underline{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{v}_3 = \begin{bmatrix} 0 \\ \frac{1}{2}(\alpha - \sqrt{1+\alpha^2} \\ 1 \end{bmatrix}$$

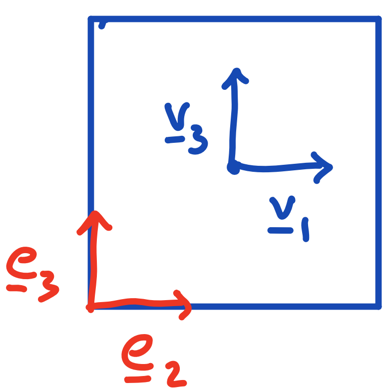
note: $|\underline{v}_1| \neq 1 \quad |\underline{v}_3| \neq 1 \quad \Rightarrow$ normalize

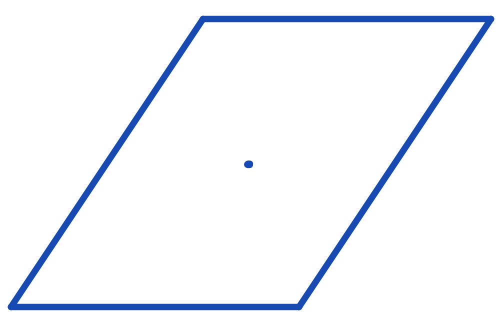
Small deformation: $\alpha \ll 1$

$$\lim_{\alpha \to 0} \underline{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \lim_{\alpha \to 0} \underline{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Large deformation: $\alpha \gg$

$$\lim_{\alpha \to \infty} \underline{v}_1 = \begin{bmatrix} 0 \\ 1 \\ \alpha \end{bmatrix} \qquad \lim_{\alpha \to \infty} \underline{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$





$$\underline{v}_1 = \frac{1}{\sqrt{1+\alpha^2}} \begin{bmatrix} 0 \\ 1 \\ \alpha \end{bmatrix}$$

$$\underline{v}_3 = \frac{1}{\sqrt{1+\alpha}}$$

What are the extreme values of the strech and their directions? $\Rightarrow$ eigenvalues & vectors

$$\begin{vmatrix} 1-\lambda^2 & \alpha & 0 \\ \alpha & 1+\alpha^2-\lambda^2 & 0 \\ 0 & 0 & 1-\lambda^2 \end{vmatrix} = 0 \qquad \begin{aligned} \lambda_1^2 &= 1+\frac{\alpha^2}{2}+\alpha\sqrt{1+\alpha^2/4} > 1 \\ \lambda_2^2 &= 1 \\ \lambda_3^2 &= 1+\frac{\alpha^2}{2}-\alpha\sqrt{1+\alpha^2/4} < 1 \end{aligned}$$

Principal directions:

$[\underline{v}_1] = [\sqrt{1+\alpha^2/4}-\alpha/2,\ 1,\ 0]$

$[\underline{v}_2] = [0,\ 0,\ 1]$ (not normalized)

$[\underline{v}_3] = [\sqrt{1+\alpha^2/4}+\alpha/2,\ -1,\ 0]$

$\Rightarrow$ λ_1 is max strech in dir $\underline{v}_1$

λ_3 is min strech in dir $\underline{v}_3$

there is no strech in dir $\underline{e}_3$