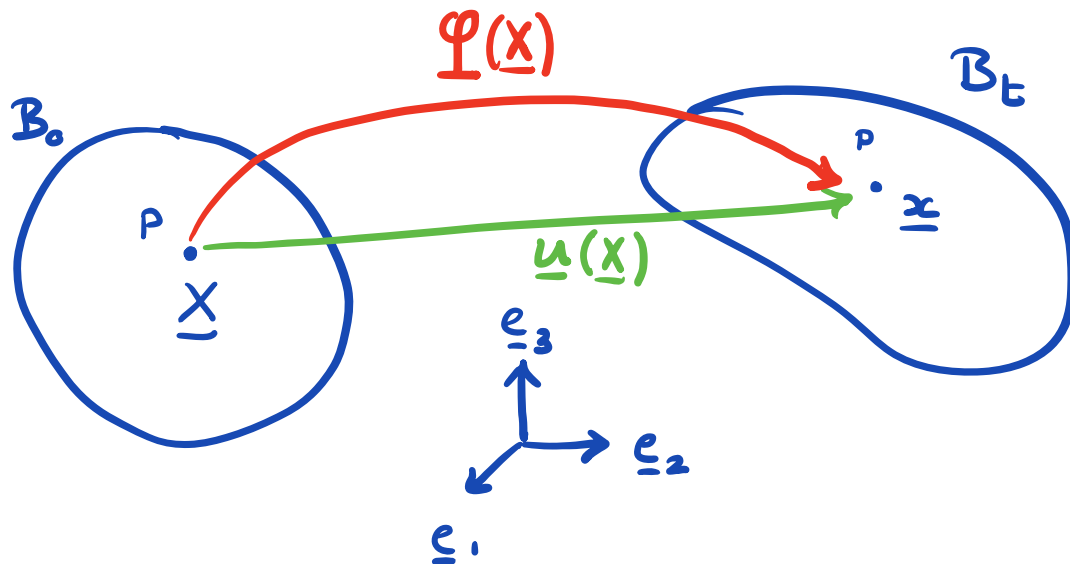


Kinematics

Study of geometry of motion without consideration of mass or stress.

⇒ Quantify the strain and rate of strain.

Deformation Mapping



B_0 = body in reference, initial, undeformed or material configuration

B_t = body in current, spatial or deformed config.

p = material point in body

\underline{X} = location of p in B_0

\underline{x} = location of p in B_t

$\underline{\varphi}(\underline{x})$ = deformation mapping

$\underline{u}(\underline{x})$ = displacement

$\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ = frame

$\underline{X} = X_I \underline{e}_I$ X_I = components of \underline{X} in $\{\underline{e}_I\}$

$\underline{x} = x_i \underline{e}_i$ $x_i =$ " " \underline{x} in $\{\underline{e}_i\}$

Convention:

Upper case quantities & indices \rightarrow reference. B_0

Lower case quantities & indices \rightarrow current. B_t

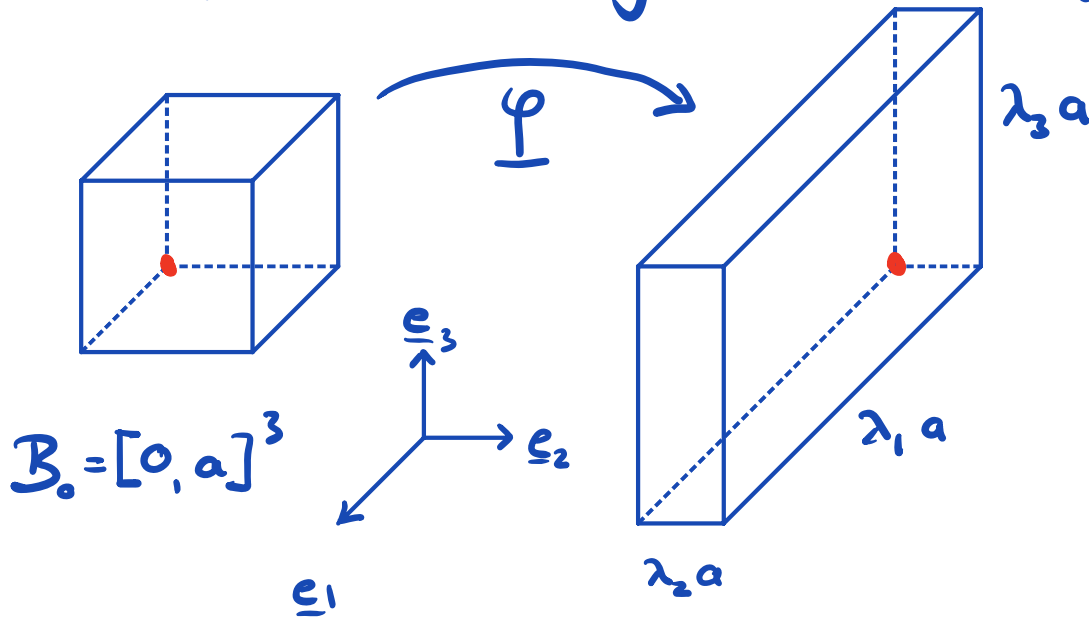
Definition of deformation mapping

$$\underline{x} = \underline{\varphi}(\underline{x}) = \varphi_i(\underline{x}) \underline{e}_i$$

Displacement of a material particle

$$\underline{u}(\underline{x}) = \underline{\varphi}(\underline{x}) - \underline{x}$$

Example: Stretching cube with edge length a



deformation map: $x_1 = \lambda_1 X_1 + v_1$

$$x_2 = \lambda_2 X_2 + v_2$$

$$x_3 = \lambda_3 X_3 + v_3$$

λ = stretch ratio

\underline{v} = translation (only important in presence of body force)

$$(\underline{v} = 0)$$

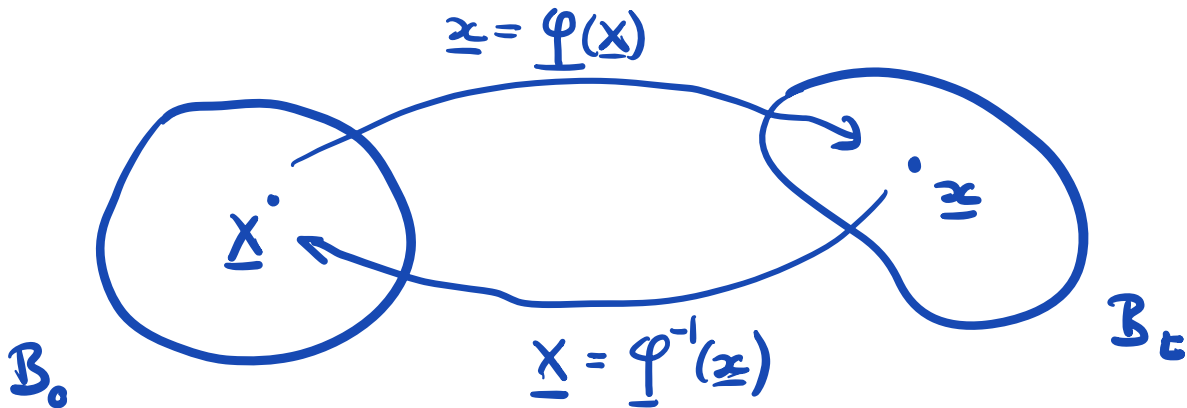
$$\underline{x} = \underline{\varphi}(\underline{X}) = \lambda_1 X_1 \underline{e}_1 + \lambda_2 X_2 \underline{e}_2 + \lambda_3 X_3 \underline{e}_3 = \underline{\Lambda}_{ij} X_j \underline{e}_i$$

$$\underline{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$\underline{x} = \underline{\Lambda} \underline{X}$$

Inverse Mapping

If φ is admissible \Rightarrow well defined inverse φ^{-1}





inverse deformation map:

$$\underline{x} = \varphi^{-1}(\underline{z})$$

Measures of Strain

In 1D we have simple measures

original:  $\Delta L = l - L$

deformed: 

engineering strain: $e = \frac{\Delta L}{L} = \frac{l - L}{L}$

stretch ratio: $\lambda = \frac{l}{L} \Rightarrow e = \lambda - 1$

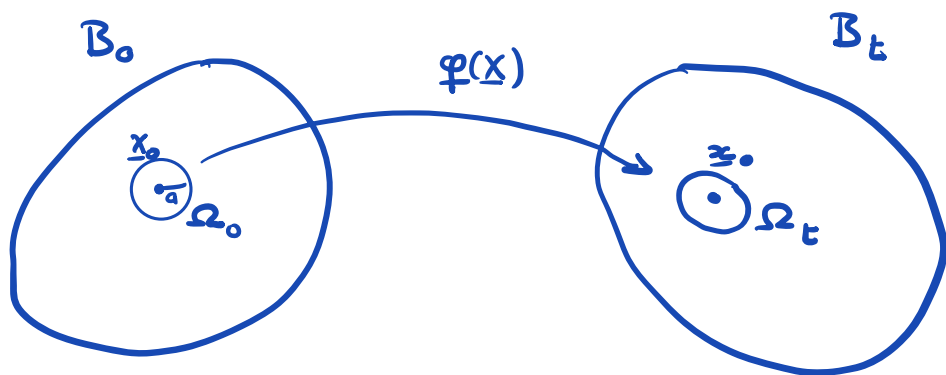
true or Hencky strain: $\epsilon = \ln(\lambda)$

Green strain: $\epsilon = \frac{1}{2}(\lambda^2 - 1)$

....

Description of strain is not unique!

Here we need to find a general 3D approach that is not limited to small deformations.



Sphere Ω_0 of radius a around \underline{x}_0 .

Mapped to Ω_t around \underline{x}_t by $\varphi(\underline{x})$

$$\Omega_t = \{ \underline{x} \in B_t \mid \underline{x} = \varphi(\underline{x}), \underline{x} \in \Omega_0 \} \rightarrow \Omega_t = \varphi(\Omega_0)$$

Def: The strain at \underline{x}_0 is any relative difference between Ω_0 and Ω_t in limit of $a \rightarrow 0$.

Deformation gradient

Natural way to quantify local strain

$$\underline{\underline{F}}(\underline{x}) = \nabla \varphi(\underline{x})$$

$$F_{ij} = \frac{\partial \varphi_i}{\partial x_j}$$

Expanding deformation in Taylor series around \underline{x}_0 we have

$$\begin{aligned} \varphi(\underline{x}) &= \varphi(\underline{x}_0) + \nabla \varphi(\underline{x}_0) (\underline{x} - \underline{x}_0) + \mathcal{O}(|\underline{x} - \underline{x}_0|^2) \\ &= \underbrace{\varphi(\underline{x}_0) - \nabla \varphi(\underline{x}_0) \underline{x}_0}_{\underline{c}} + \underbrace{\nabla \varphi(\underline{x}_0)}_{\underline{\underline{F}}(\underline{x}_0)} \underline{x} \end{aligned}$$

locally we can approximate φ as

$$\varphi(\underline{x}) \approx \underline{c} + \underline{\underline{F}}(\underline{x}_0) \underline{x} \quad (\text{affine deform.})$$

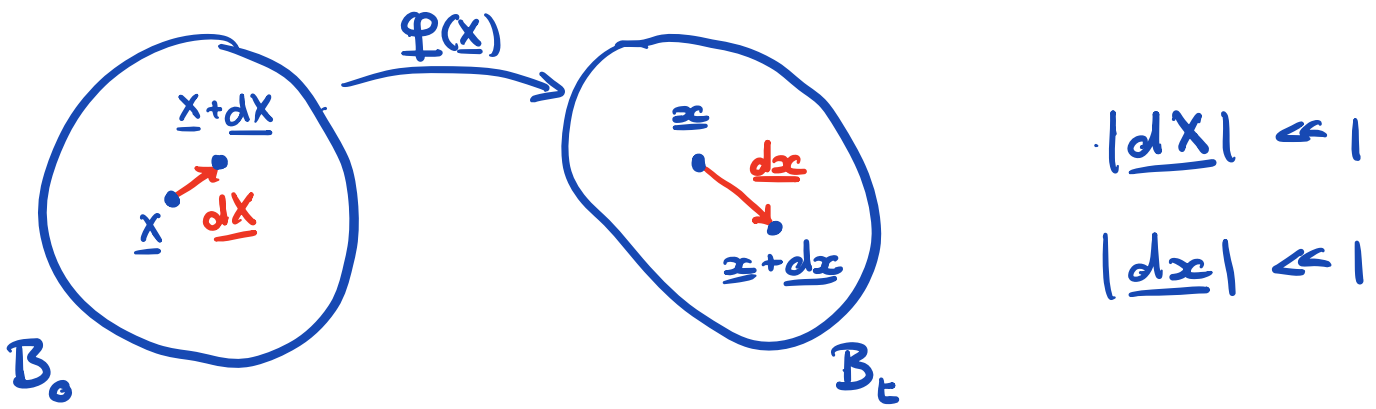
$\Rightarrow \underline{\underline{F}}(\underline{x}_0)$ characterizes local behavior of $\varphi(\underline{x})$

Homogeneous deformation

$\underline{\underline{F}}$ is constant

$$\Rightarrow \underline{x} = \varphi(\underline{x}) = \underline{c} + \underline{\underline{F}} \underline{x}$$

Consider the mapping of line segment



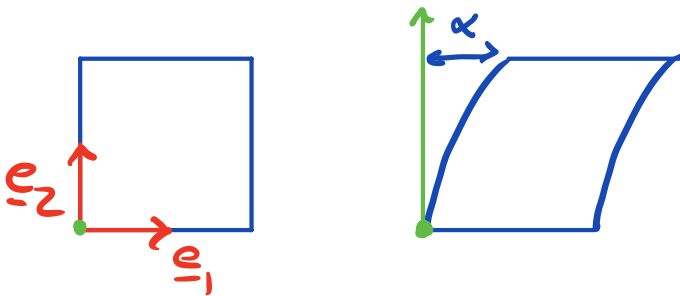
$$\underline{x} + d\underline{x} = \varphi(\underline{x} + d\underline{X}) \approx \varphi(\underline{x}) + \nabla \varphi(\underline{x}) d\underline{X} = \underline{x} + \underline{\underline{F}}(\underline{x}) d\underline{X}$$

$$d\underline{x} = \underline{\underline{F}}(\underline{x}) d\underline{X}$$

$$dx_i = F_{ij}(\underline{x}) dX_j$$

$\underline{\underline{F}}$ maps material vectors into spatial vectors.

Example: Shear deformation



$$\varphi(\underline{x}) = [x_1 + \alpha x_2^2, x_2]$$

$$\nabla \varphi = \underline{\underline{F}} = \begin{bmatrix} 1 & 2\alpha x_2 \\ 0 & 1 \end{bmatrix}$$

$$\underline{\underline{F}} \underline{e}_1 = [1, 0]^T = \underline{e}_1$$

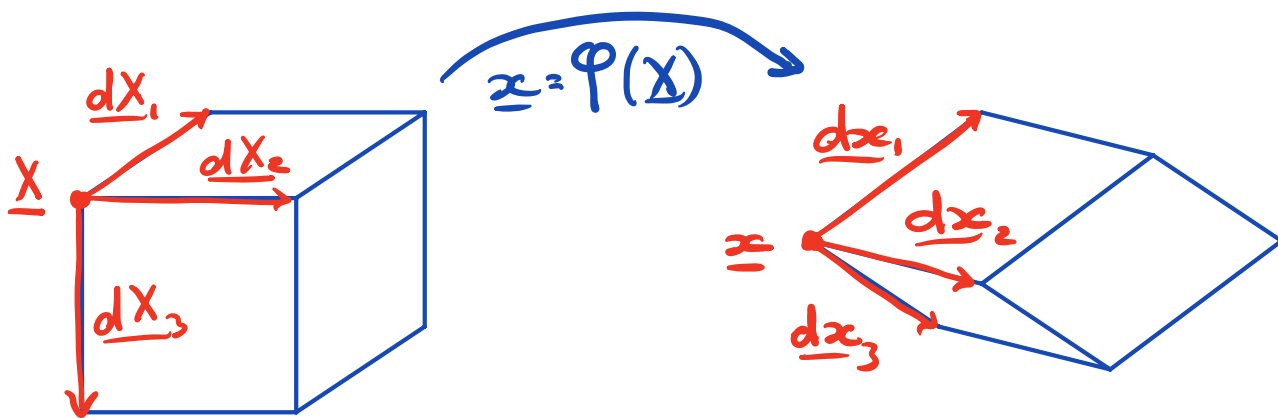
unchanged

$$\underline{\underline{F}} \underline{e}_2 = [2\alpha x_2, 1]^T$$

rotated and stretched

Volume changes

Change in volume during deformation



$$\text{Volumes are: } dV_X = (d\underline{X}_1 \times d\underline{X}_2) \cdot d\underline{X}_3$$

$$\begin{aligned} dV_x &= (d\underline{x}_1 \times d\underline{x}_2) \cdot d\underline{x}_3 \\ &= \det([d\underline{x}_1][d\underline{x}_2][d\underline{x}_3]) \end{aligned}$$

substituting $d\underline{x} = \underline{F} d\underline{X}$

$$dV_x = \det([\underline{F} d\underline{X}_1] [\underline{F} d\underline{X}_2] [\underline{F} d\underline{X}_3])$$

$$= \det(\underline{F} \underline{dX}) \quad \text{where } \underline{dX} = [d\underline{X}_1 \ d\underline{X}_2 \ d\underline{X}_3]$$

$$= \det(\underline{F}) \det(\underline{dX})$$

$$= \det(\underline{F}) (d\underline{X}_1 \times d\underline{X}_2) \cdot d\underline{X}_3$$

$$\Rightarrow \boxed{dV_x = \det(\underline{F}) dV_X}$$

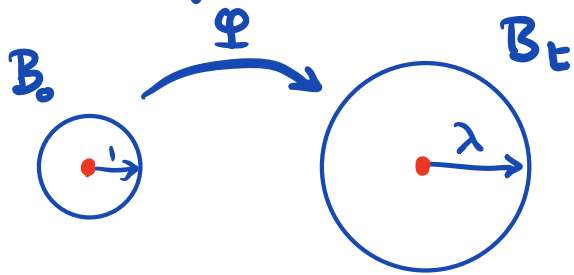
The field $\underline{J}(\underline{x}) = \det(\underline{F}) = \frac{dV_{\infty}}{dV_{\underline{x}}}$ is the Jacobian of φ and measures the volume strain.

$\underline{J}(\underline{x}) > 1$: volume increase

$\underline{J}(\underline{x}) < 1$: volume decrease

$\underline{J}(\underline{x}) = 1$: no volume change

Example: Expanding sphere $V = \frac{4}{3}\pi R^3$



$$V_0 = \frac{4\pi}{3}$$

$$V_t = \frac{4\pi}{3} \lambda^3$$

Deformation map: $\underline{x} = \varphi(\underline{X}) = \lambda \underline{X} \quad \lambda > 1$

$$\underline{F} = \nabla \varphi = \lambda \underline{I}$$

$\underline{J} \neq \underline{J}(\underline{x})$ because \underline{F} is const

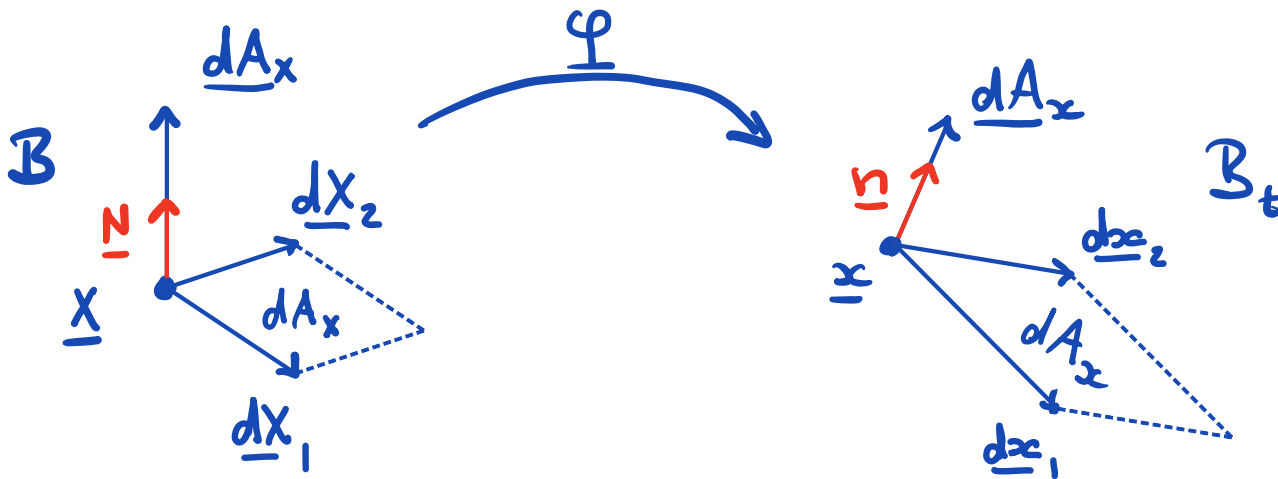
$$\underline{J} = \det(\underline{F}) = \det(\lambda \underline{I}) = \lambda^3 \underbrace{\det(\underline{I})}_{1}$$

$$\underline{J} = \lambda^3$$

$$V_t = \underline{J} V_0 = \frac{4\pi}{3} \lambda^3 \quad \checkmark$$

Surface area changes

How do surfaces change during deformation



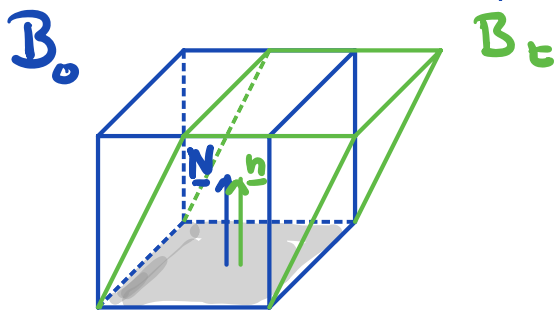
surface normals: $|\underline{N}| = |\underline{n}| = 1$

surface vector elements: $\underline{dA}_x = \underline{N} dA_x = \underline{dX}_1 \times \underline{dX}_2$

$\underline{dA}_x = \underline{n} dA_x = \underline{dx}_1 \times \underline{dx}_2$

Important: $\underline{n} \neq \underline{F} \underline{N}$!

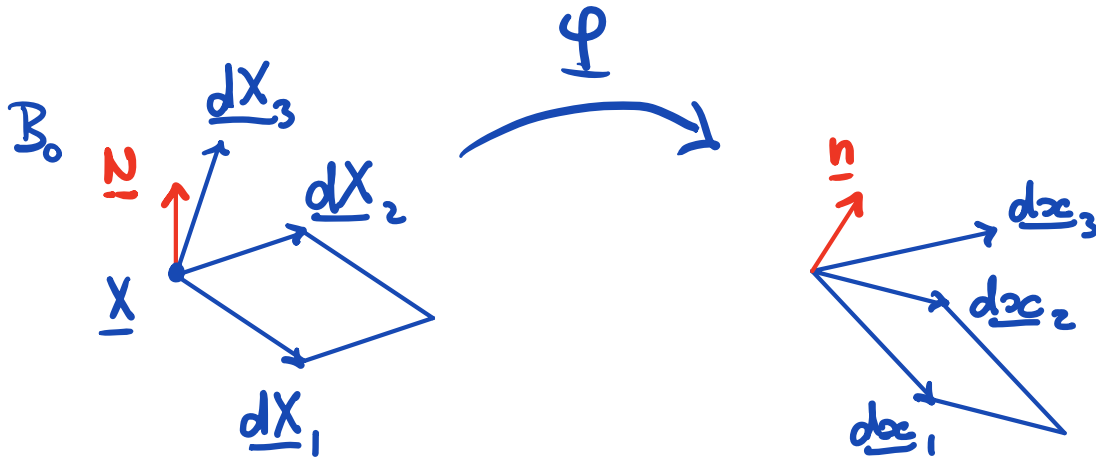
Example: Simple shear



$$\underline{N} = \underline{n} \neq \underline{F} \underline{N}$$

What is the relation between \underline{N} and \underline{n} ?

Consider \underline{dX}_3 so that $\underline{N} \cdot \underline{dX}_3 \neq 0$



$$\underline{dA}_x = \underline{dX}_1 \times \underline{dX}_2$$

$$dV_x = \underline{dA}_x \cdot \underline{dX}_3$$

$$\underline{dA}_x = \underline{dx}_1 \times \underline{dx}_2$$

$$dV_x = \underline{dA}_x \cdot \underline{dx}_3$$

Change in volume:

$$dV_x = J dV_x$$

$$\underline{dA}_x \cdot \underline{dx}_1 = J \underline{dA}_x \cdot \underline{dX}_1$$

with $\underline{dx}_1 = \underline{F} \underline{dX}_1$

$$\underline{dA}_x \cdot \underline{F} \underline{dX}_1 - J \underline{dA}_x \cdot \underline{dX}_1 = 0$$

using transpose

$$\underline{F}^T \underline{dA}_x \cdot \underline{dX}_1 - J \underline{dA}_x \cdot \underline{dX}_1 = 0$$

$$(\underline{F}^T)^{-1} (\underline{F}^T \underline{dA}_x - J \underline{dA}_x) \cdot \underline{dX}_1 = 0$$

since \underline{dX} is arbitrary

\Rightarrow

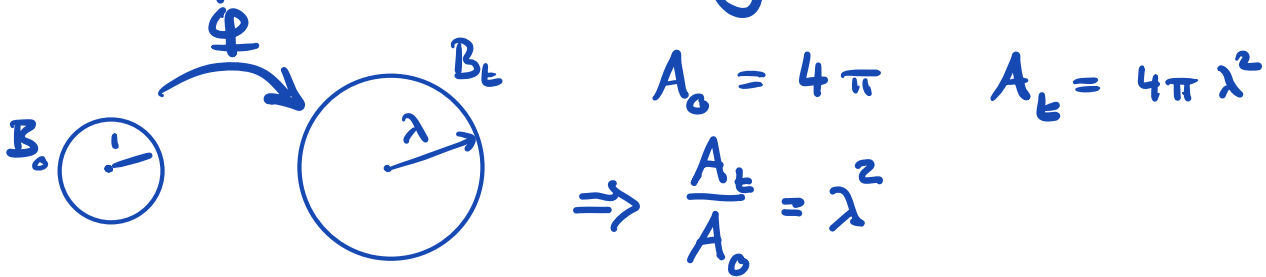
$$\underline{dA}_x = J \underline{F}^{-T} \underline{dA}_x$$

$$\underline{n} dA_x = J \underline{F}^{-T} \underline{N} dA_x$$

Nanson's formula
 $\underline{F}^{-T} = (\underline{F}^T)^{-1}$

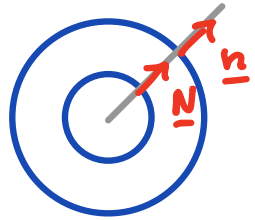
so that $\underline{n} = \frac{\mathcal{J} dA_x}{dA_x} \underbrace{\underline{F}^{-T} \underline{N}}_{\text{normalization direction}}$

Example: Expanding sphere



Get same result with Nausen's formula:

Both B_0 & B_t are spheres: $\underline{N} = \underline{n}$!

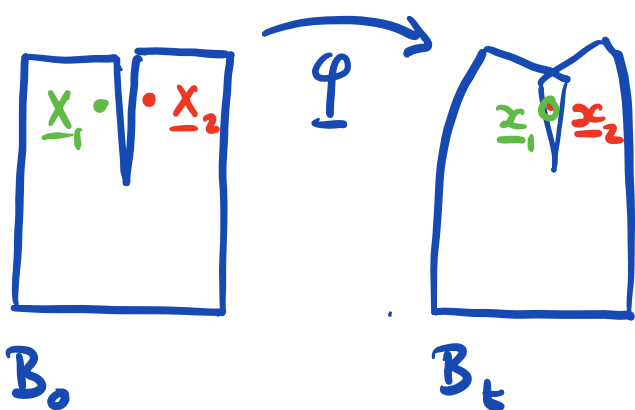


$\underline{n} dA_x = \mathcal{J} \underline{F}^{-T} \underline{N} dA_x$ $\underline{n} \frac{dA_x}{dA_x} = \mathcal{J} \underline{F}^{-T} \underline{N}$ <p>substitute \mathcal{J} & \underline{F}^{-T}</p> $\underline{n} \frac{dA_x}{dA_x} = \lambda^3 \frac{1}{\lambda} \underline{I} \underline{N}$ $\underline{n} \frac{dA_x}{dA_x} = \lambda^2 \underline{N} \quad \text{since } \underline{n} = \underline{N}$ $\cancel{\underline{n}} \cdot \cancel{\underline{n}'} \frac{dA_x}{dA_x} = \lambda^2 \cancel{\underline{n}} \cdot \cancel{\underline{n}'} \Rightarrow \frac{dA_x}{dA_x} = \lambda^2 \quad \checkmark$	<p>From volume change example</p> $\varphi = \lambda \underline{x} \quad \& \quad \underline{F} = \lambda \underline{I}$ $\mathcal{J} = \det(\underline{F}) = \lambda^3$ $\Rightarrow \underline{F}^{-T} = \underline{F}^{-1} = \frac{1}{\lambda} \underline{I}$
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Admissible deformations

For φ to represent the deformation of a body it must satisfy the following conditions:

1) $\varphi: B_0 \rightarrow B_t$ is one to one and onto



two separate points in B_0 cannot be mapped to same point in B_t .

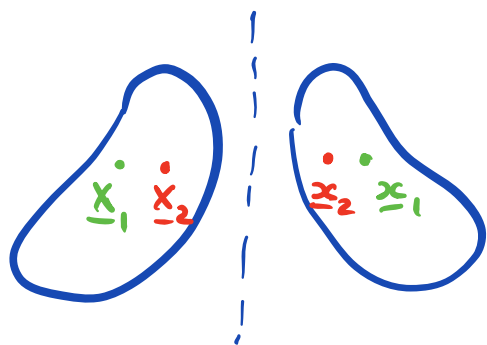
one to one: for each \underline{x} in B_0 there is at most one

$$\underline{x} \text{ in } B_t \text{ s.t. } \underline{x} = \varphi(\underline{x})$$

onto: for each \underline{x} in B_0 there is at least one

$$\underline{x} \text{ in } B_t \text{ s.t. } \underline{x} = \varphi(\underline{x})$$

2) $\det(\nabla\varphi) > 0$



The orientation of a body is preserved, i.e., a body cannot be deformed into its mirror image.

Next time: Analysis of local deformation
series of decompositions

I) Translation - Fixed point decomposition

$\varphi(\underline{x}) \rightarrow$ translation & def. with fixed point

II) Polar decomposition

def with fixed point \rightarrow rotation & stretch

III) Spectral decomposition

stretch \rightarrow principal stretches

\Rightarrow allows us to formulate strain tensors