Differentiation of Tensor fields

A field is a function of space.

scalar fields: $\phi(\underline{x})$ temp., density

vector fields: $\underline{v}(\underline{x})$ force, velocity

tensor fields: $\underline{\underline{S}}(\underline{x})$ stress, conductivity

Today's lecture is review and extension of concepts from multivariable calculus.

Gradients

Gradient of scalar field

Scalar field $\phi(\underline{x})$ is differentiable at $\underline{x}$ if there exists a vector field $\nabla\phi \in \mathcal{V}$ such that

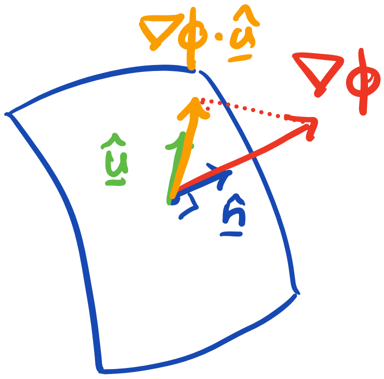
$$\boxed{\phi(\underline{x}+\underline{h}) = \phi(\underline{x}) + \nabla\phi(\underline{x})\cdot\underline{h} + \mathcal{O}(|\underline{h}|)}$$

by Taylor expansion. Or equivalently

$$\boxed{\nabla\phi(\underline{x})\cdot\underline{\hat{u}} = \frac{d}{d\varepsilon}\phi(\underline{x}+\epsilon\underline{\hat{u}})\Big|_{\epsilon=0}} \quad \text{for all } \underline{v}\in\mathcal{V}$$

where $\underline{h} = \epsilon\,\underline{\hat{u}}$ and $|\underline{\hat{u}}| = 1$.

The vector $\nabla\phi$ is called the gradient of ϕ.



Consider a level set of ϕ

$\nabla\phi \parallel \hat{\underline{n}}$ in direction of increasing ϕ

$$\hat{\underline{n}} = \frac{\nabla\phi}{|\nabla\phi|}$$

Directional derivative (Gâteaux operator)

$$\boxed{D_{\hat{\underline{u}}}\phi(\underline{x}) = \frac{d}{d\epsilon}\phi(\underline{x}+\epsilon\,\hat{\underline{u}})\Big|_{\epsilon=0} = \nabla\phi(\underline{x})\cdot\hat{\underline{u}}}$$

Representation of the gradient in frame $\{\underline{e}_i\}$

$$\phi(\bar{\underline{x}}+\epsilon\,\underline{u}) = \phi(\underbrace{\bar{x}_1+\epsilon u_1}_{x_1}, \underbrace{\bar{x}_2+\epsilon u_2}_{x_2}, \underbrace{\bar{x}_3+\epsilon u_3}_{x_3})$$

$$\nabla\phi\cdot\hat{\underline{u}} = \frac{d}{d\epsilon}\phi(\bar{x}_1+\epsilon u_1, \bar{x}_2+\epsilon u_2, \bar{x}_3+\epsilon u_3)\Big|_{\epsilon=0}$$

$$= \frac{d\phi}{dx_1}\frac{dx_1}{d\epsilon} + \frac{d\phi}{dx_2}\frac{dx_2}{d\epsilon} + \frac{d\phi}{dx_3}\frac{dx_3}{d\epsilon}\Big|_{\epsilon=0}$$

$$= \frac{d\phi}{dx_1}u_1 + \frac{d\phi}{dx_2}u_2 + \frac{d\phi}{dx_3}u_3$$

$$= \frac{\partial\phi}{\partial x_i}u_i = \phi_{,i}u_i = \phi_{,i}u_j\delta_{ij} = \phi_{,i}u_j\,\underline{e}_i\cdot\underline{e}_j$$

$$\nabla\phi\cdot\hat{\underline{u}} = (\phi_{,i}\,\underline{e}_i)\cdot(u_j\,\underline{e}_j) \quad \checkmark$$

Note: Index notation for derivatives

$$\boxed{\frac{\partial \phi}{\partial x_i} = \phi_{,i}}$$ derivative index after comma!

Gradient in components: $[\nabla \phi] = \phi_{,i} \underline{e}_i = \begin{pmatrix} \frac{\partial \phi}{\partial x_1} \\ \frac{\partial \phi}{\partial x_2} \\ \frac{\partial \phi}{\partial x_3} \end{pmatrix}$

<u>Heat conduction</u>:

potential: $T(\underline{x})$ Temperature

heat flux: $\boxed{\underline{q} = -k \nabla T}$ (Fourier's law)

k: thermal conductivity $\left[\frac{W}{mK}\right]$

Molecular diffusion:

potential: $c(\underline{x})$ Concentration

diffusive flux: $\boxed{\underline{j} = -D \nabla c}$ (Fick's law)

D: molecular diffusivity $\left[\frac{m^2}{s}\right]$

Groundwater flow:

potential: $h(\underline{x})$ hydraulic head

volumetric flux: $\boxed{\underline{q} = -K \nabla h}$ (Darcy's law)

K: hydraulic conductivity $\left[\frac{m}{s}\right]$

Gravitational field: $\underline{g} = -\nabla \Phi$

Gradient of a vector field

A vector field $\underline{v}(x) \in \mathcal{V}$ is differentiable at $\underline{x}$ if there exists a tensor field $\nabla \underline{v}(\underline{x}) \in \mathcal{V}^2$ such that

$$\underline{v}(\underline{x}+\underline{h}) = \underline{v}(\underline{x}) + \nabla \underline{v}(x)\, \underline{h} + o(|h|)$$

by Taylor expansion or equivalently

$$\boxed{\nabla \underline{v}\, \hat{\underline{u}} = \frac{d}{d\epsilon} \underline{v}(\underline{x}+\epsilon \hat{\underline{u}})\Big|_{\epsilon=0}} \quad \text{for all } \underline{u} \in \mathcal{V}$$

where $\underline{h} = \epsilon \hat{\underline{u}}$

In frame $\{\underline{e}_i\}$ we write components of $\underline{v}$ as $v_i = v_i(x_1, x_2, x_3)$. For any scalar ϵ and unit vector $\hat{\underline{u}} = u_k \underline{e}_k$ at $\bar{\underline{x}} = \bar{x}_j \underline{e}_j$ we have the i-th component

$$v_i(\bar{\underline{x}} + \epsilon \hat{\underline{u}}) = v_i(\bar{x}_1 + \epsilon u_1, \bar{x}_2 + \epsilon u_2, \bar{x} + \epsilon u_3)$$

by the chain rule

$$\frac{d}{d\epsilon} v_i(\bar{\underline{x}} + \epsilon \hat{\underline{u}}) = \frac{\partial v_i}{\partial x_1} u_1 + \frac{\partial v_i}{\partial x_2} u_2 + \frac{\partial v_i}{\partial x_3} u_3 = \frac{\partial v_i}{\partial x_j} u_j$$

For full vector $\underline{v} = v_i \underline{e}_i$

$$\nabla \underline{v}\, \hat{\underline{u}} = \frac{d}{d\varepsilon} \underline{v}(\bar{\underline{x}} + \epsilon \hat{\underline{u}})\Big|_{\epsilon=0} = \frac{d}{d\varepsilon} \left(v_i(\bar{\underline{x}} + \epsilon \hat{\underline{u}}) \underline{e}_i \right)\Big|_{\epsilon=0}$$

$$= \frac{d}{d\varepsilon} \left(v_i(\bar{\underline{x}} + \epsilon \hat{\underline{u}}) \right)\Big|_{\epsilon=0} \underline{e}_i = \frac{\partial v_i}{\partial x_j} u_j \underline{e}_i$$

components: $\boxed{[\nabla \underline{v}]_{ij} = \frac{\partial v_i}{\partial x_j} = v_{i,j}}$

Representation $\nabla \underline{v} = v_{i,j} \underline{e}_i \otimes \underline{e}_j$

Explicit

$$\nabla \underline{v} = \begin{bmatrix} v_{1,1} & v_{1,2} & v_{1,3} \\ v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,1} & v_{3,2} & v_{3,3} \end{bmatrix} = \begin{bmatrix} \nabla v_1^T \\ \nabla v_2^T \\ \nabla v_3^T \end{bmatrix}$$

$$= \left[\frac{\partial \underline{v}}{\partial x_1} \; \frac{\partial \underline{v}}{\partial x_2} \; \frac{\partial \underline{v}}{\partial x_3} \right]$$

Examples:

1) Strain tensor : $\underline{\underline{\varepsilon}} = \text{sym}(\nabla \underline{u}) = \frac{1}{2}(\nabla \underline{u} + \nabla \underline{u}^T)$

2) Rate of strain tensor: $\dot{\underline{\underline{\varepsilon}}} = \frac{1}{2}(\nabla \underline{v} + \nabla \underline{v}^T)$

$\underline{u}$ = displacement $\underline{v}$ = velocity

Divergence of a vector field

Def: To any $\underline{v}(x) \in \mathcal{V}$ we associate a scalar field $\nabla \cdot \underline{v}$ called the divergence of $\underline{v}$

$$\boxed{\nabla \cdot \underline{v} = tr(\nabla \underline{v})}$$

In frame $\{\underline{e}_i\}$ with $\underline{v}(\underline{x}) = v_i(\underline{x})\,\underline{e}_i$ we have

$$\boxed{\nabla \cdot \underline{v} = tr(\nabla \underline{v}) = v_{i,i}}$$

If $\nabla \cdot \underline{v} = 0$ a field is solenoidal or divergence free. If $\underline{v}$ is a displacement or velocity then $\nabla \cdot \underline{v}$ is related to (rate of) volume change.

Examples:

Gauss' law of gravity: $\nabla \cdot g = -4\pi G \rho$

Continuity condition: $\nabla \cdot \underline{v} = 0$

(incompressible flows)

Divergence of a tensor field

To any $\underline{\underline{S}}(\underline{x}) \in \mathcal{V}^2$ we associate a vector field $\nabla \cdot \underline{\underline{S}} \in \mathcal{V}$ called the divergence of $\underline{\underline{S}}$

$$\boxed{(\nabla \cdot \underline{\underline{S}}) \cdot \underline{a} = \nabla \cdot (\underline{\underline{S}}^T \underline{a})} \quad \text{for all } \underline{a} \in \mathcal{V}$$

uses definition of vector divergence!

In frame $\{\underline{e}_i\}$ with $\underline{\underline{S}} = S_{ij}\, \underline{e}_i \otimes \underline{e}_j$ and $\underline{a} = a_k \underline{e}_k$

we have $\underline{q} = \underline{\underline{S}}^T \underline{a}$ or $q_j = S_{ij} a_i$ $(q_i = S_{ji} a_j)$

substituting

$$(\nabla \cdot \underline{\underline{S}}) \underline{a} = \nabla \cdot (\underline{\underline{S}}^T \underline{a}) = \nabla \cdot \underline{q} = \operatorname{tr}(\nabla \underline{q}) = q_{j,j}$$

$$= S_{ij,j}\, a_i = (S_{ij,j}\, \underline{e}_i) \cdot (a_k \underline{e}_k)$$

by the arbitraryness of $\underline{a}$ we have

$$\boxed{\nabla \cdot \underline{\underline{S}} = S_{ij,j}\, \underline{e}_i}$$

Gradient & Divergence product rules

$$\boxed{\begin{aligned}
\nabla\cdot(\phi\,\underline{v}) &= \underline{v}\cdot\nabla\phi + \phi\,\nabla\cdot\underline{v} \\
\nabla\cdot(\phi\,\underline{\underline{S}}) &= \underline{\underline{S}}\,\nabla\phi + \phi\,\nabla\cdot\underline{\underline{S}} \\
\nabla\cdot(\underline{\underline{S}}^T\underline{v}) &= (\nabla\cdot\underline{\underline{S}})\cdot\underline{v} + \underline{\underline{S}}:\nabla\underline{v} \\
\nabla(\phi\,\underline{v}) &= \underline{v}\otimes\nabla\phi + \phi\,\nabla\underline{v}
\end{aligned}}$$

$\phi\in\mathbb{R}$

$\underline{v}\in\mathcal{V}$,

$\underline{\underline{S}}\in\mathcal{V}^2$

Example: $\nabla\cdot(\underline{\underline{S}}^T\underline{v})$ note $\underline{\underline{S}}=\underline{\underline{S}}(x)$ and $\underline{v}=v(\underline{x})$

$$q(x) = \underline{\underline{S}}^T(\underline{x})\,\underline{v}(\underline{x}) \qquad q_j = S_{ij}v_i$$

$$\begin{aligned}
\nabla\cdot q = tr(q) = q_{j,j} &= (S_{ij}v_i)_{,j} \\
&= S_{ij,j}\,v_i + S_{ij}\,v_{i,j} \\
&= (\nabla\cdot\underline{\underline{S}})\cdot\underline{v} + \underline{\underline{S}}:\nabla\underline{v} \quad \checkmark
\end{aligned}$$

$\rightarrow$ useful for energy balance!

Example:
$$\begin{aligned}
\nabla(\phi\,\underline{v}) &= (\phi\,v_i)_{,j}\,\underline{e}_i\otimes\underline{e}_j \\
&= (\phi_{,j}v_i + \phi\,v_{i,j})\,\underline{e}_i\otimes\underline{e}_j \\
&= v_i\,\phi_{,j}\,\underline{e}_i\otimes\underline{e}_j + \phi\,v_{i,j}\,\underline{e}_i\otimes\underline{e}_j \\
&= \underline{v}\otimes\nabla\phi + \phi\,\nabla\underline{v} \quad \checkmark
\end{aligned}$$

Laplacian

I) Laplacian of scalar field

$$\boxed{\Delta \phi = \nabla^2 \phi = \nabla \cdot \nabla \phi}$$

In frame $\{\underline{e}_i\}$ with $\nabla \phi = \phi_{,i} \underline{e}_i$ we have

$$\nabla \cdot \nabla \phi = \mathrm{tr}(\nabla \nabla \phi) = \mathrm{tr}(\phi_{,ij} \underline{e}_i \otimes \underline{e}_j) = \phi_{,ii}$$

$$\boxed{\nabla^2 \phi = \phi_{,ii}}$$

Scalar Laplacian governs steady heat flow.

Example: Poisson's equation for gravity

1) Grav. field eqn: $\quad \underline{g} = -\nabla \Phi$

2) Gauss' law : $\quad \nabla \cdot \underline{g} = -4\pi G \rho$

$$-\nabla \cdot \nabla \Phi = -4\pi G \rho$$

$$\boxed{\nabla^2 \Phi = 4\pi G \rho}$$

II Laplacian of vector field

$$\boxed{\Delta \underline{v} = \nabla^2 \underline{v} = \nabla \cdot \nabla \underline{v}}$$

in index notation:

$\underline{v} = v_i \underline{e}_i$, $\nabla \underline{v} = v_{i,j}\, \underline{e}_i \otimes \underline{e}_j$ and $\nabla \cdot \underline{\underline{S}} = S_{ij,j}\, \underline{e}_i$

$$\boxed{\Delta \underline{v} = v_{i,jj}\, \underline{e}_i}$$

Example: Stokes flow (creeping flow)

$$\mu \nabla^2 \underline{v} = \nabla p \qquad \mu = \text{viscosity}$$

$$\nabla \cdot \underline{u} = 0 \qquad p = \text{pressure}$$