Green-Lagrange Strain Tensor

right Cauchy-Green: $\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}}$

$\Rightarrow$ quantifies streches (& shear)

$\lambda(\hat{\underline{X}}) = |d\underline{x}| / |d\underline{X}| = \sqrt{\hat{\underline{X}} \cdot \underline{\underline{C}}\, \hat{\underline{X}}}$

No deformation: $\underline{x} = \varphi(\underline{X}) = \underline{X} \Rightarrow \underline{\underline{F}} = \underline{\underline{C}} = \underline{\underline{I}} \quad \Rightarrow \lambda = 1$

Constitutive law: $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}(\underline{\underline{C}})$

no deformation = no stress

$\Rightarrow$ Strain tensor zero in absence of deformation

1D: engineering strain: $e = \lambda - 1$

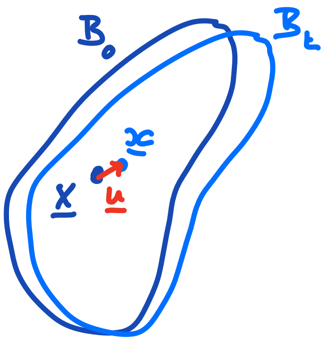
3D Green-Lagrange strain tensor

$$\boxed{\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}})}$$

no deformation: $\underline{\underline{F}} = \underline{\underline{C}} = \underline{\underline{I}} \quad \Rightarrow \quad \underline{\underline{E}} = \underline{\underline{0}}$

relation to infinitesimal strain tensor

Small displacements



natural to use displacement

$$\boxed{\underline{u} = \underline{x} - \underline{X}} \qquad \underline{x} = \varphi(\underline{X})$$

$$\nabla \underline{u} = \nabla(\varphi(\underline{X}) - \underline{X}) = \nabla\varphi - \underline{\underline{I}}$$

$$\nabla \underline{u} = \underline{\underline{F}} - \underline{\underline{I}} \equiv \underline{\underline{H}}$$

Quantify magnitude of tensor:

$$|\underline{\underline{A}}| = \sqrt{\underline{\underline{A}} : \underline{\underline{A}}} = (A_{11}^2 + A_{12}^2 + \ldots + A_{32}^2 + A_{33}^2)^{\frac{1}{2}}$$

Small deformation: $|\underline{\underline{H}}| = \epsilon \ll 1$

Linearize Cauchy-Green

$$\underline{\underline{F}} = \underline{\underline{I}} + \underline{\underline{H}} \qquad |\underline{\underline{H}}| = \epsilon \ll 1$$

$$\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}} = (\underline{\underline{I}} + \underline{\underline{H}})^T (\underline{\underline{I}} + \underline{\underline{H}})$$

$$= (\underline{\underline{I}} + \underline{\underline{H}}^T)(\underline{\underline{I}} + \underline{\underline{H}})$$

$$= \underline{\underline{I}} + \underline{\underline{H}} + \underline{\underline{H}}^T + \underbrace{\underline{\underline{H}}^T \underline{\underline{H}}}_{O(\epsilon^2)}$$

$$\Rightarrow \quad \underline{\underline{C}} \approx \underline{\underline{I}} + \underline{\underline{H}} + \underline{\underline{H}}^T = \underline{\underline{I}} + \nabla \underline{u} + \nabla \underline{u}^T$$

Linearize Euler-Lagrange

$$\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}}) = \frac{1}{2}(\underline{\underline{I}} + \nabla\underline{u} + \nabla\underline{u}^T - \underline{\underline{I}}) + O(|\nabla u|^2)$$

$$E \approx \frac{1}{2}(\nabla\underline{u} + \nabla\underline{u}^T)$$

Infinitesimal strain tensor:

$$\boxed{\underline{\underline{\varepsilon}} = \frac{1}{2}(\nabla\underline{u} + \nabla\underline{u}^T)} \qquad \underline{\underline{\varepsilon}} = \mathrm{sym}(\nabla\underline{u})$$

Infinitesimal Stretch & Rotation

Linearize right strech:

$$\underline{\underline{U}} = \sqrt{\underline{\underline{C}}} = (\underline{\underline{I}} + \underline{\underline{H}} + \underline{\underline{H}}^T + \underline{\underline{H}}^T\underline{\underline{H}})^{\frac{1}{2}}$$

Note: for $\underline{\underline{A}} = \underline{\underline{A}}^T$ $\quad m \in \mathbb{R}$

$$(\underline{\underline{I}} + \underline{\underline{A}})^m = \underline{\underline{I}} + m\underline{\underline{A}} + O(|\underline{\underline{A}}|^2)$$

shown with Taylor expansion in principal frame

$$\underline{\underline{A}} = H + H^T + O(|\underline{\underline{H}}|^2)$$

$$\Rightarrow \underline{\underline{U}} = \underline{\underline{I}} + \frac{1}{2}(\underline{\underline{H}} + \underline{\underline{H}}^T) + O(|\underline{\underline{H}}|^2)$$

$$\approx \underline{\underline{I}} + \frac{1}{2}(\nabla\underline{u} + \nabla\underline{u}^T) = \underline{\underline{I}} + \underline{\underline{\varepsilon}}$$

Similarly $\underline{\underline{V}} = \sqrt{\underline{\underline{F}}\,\underline{\underline{F}}^T} = \underline{\underline{I}} + \frac{1}{2}(\nabla\underline{u} + \nabla\underline{u}^T) = \underline{\underline{I}} + \underline{\underline{\varepsilon}}$

Note: $\underline{\underline{C}} \approx \underline{\underline{I}} + \underline{\underline{\varepsilon}}$, $\underline{\underline{U}} \approx \underline{\underline{I}} + \underline{\underline{\varepsilon}}$, $\underline{\underline{V}} \approx \underline{\underline{I}} + \underline{\underline{\varepsilon}}$

$\Longrightarrow$ linearization is the same

Linearize rotation

$$\underline{\underline{R}} = \underline{\underline{F}}\,\underline{\underline{U}}^{-1} = (\underline{\underline{I}} + \underline{\underline{H}})(\underline{\underline{I}} + \underline{\underline{\varepsilon}})^{-1} \qquad \underline{\underline{\varepsilon}} = O(|\underline{\underline{H}}|)$$

$$= (\underline{\underline{I}} + \underline{\underline{H}})(\underline{\underline{I}} - \underline{\underline{\varepsilon}}) + O(|\underline{\underline{H}}|^2)$$

$$= \underline{\underline{I}} - \frac{1}{2}(\underline{\underline{H}} + \underline{\underline{H}}^T) + \underline{\underline{H}} + O(|\underline{\underline{H}}|^2)$$

$$= \underline{\underline{I}} + \underline{\underline{H}} - \underline{\underline{\varepsilon}} + O(|\underline{\underline{H}}|^2)$$

$$= \underline{\underline{I}} + \frac{1}{2}(\underline{\underline{H}} - \underline{\underline{H}}^T) + O(|\underline{\underline{H}}|^2)$$

$$\underline{\underline{R}} \approx \underline{\underline{I}} + \frac{1}{2}(\nabla\underline{u} - \nabla\underline{u}^T)$$

$$= \underline{\underline{I}} + \underline{\underline{\omega}}$$

Infinitesimal Rotation Tensor: $\boxed{\underline{\underline{\omega}} = \frac{1}{2}(\nabla\underline{u} - \nabla\underline{u}^T)}$

$$\underline{\underline{\omega}} = \text{skew}(\nabla\underline{u}) = \text{axial tensor}$$

axis of rotation: $a_j = \frac{1}{2}\,\epsilon_{mjn}\,\omega_{mn}$

Linearization of $\underline{\underline{F}}$

$$\underline{\underline{U}} \approx \underline{\underline{I}} + \underline{\underline{\varepsilon}} \qquad \underline{\underline{R}} = \underline{\underline{I}} + \underline{\underline{\omega}}$$

$$\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} = (\underline{\underline{I}} + \underline{\underline{\varepsilon}})(\underline{\underline{I}} + \underline{\underline{\omega}}) = \underline{\underline{I}} + \underline{\underline{\varepsilon}} + \underline{\underline{\omega}} + O(|H^2|)$$

$$\Rightarrow \quad \boxed{\underline{\underline{F}} = \underbrace{\underline{\underline{R}}\,\underline{\underline{U}}}_{\text{multipicative}} \approx \underline{\underline{I}} + \underbrace{\underline{\underline{\varepsilon}} + \underline{\underline{\omega}}}_{\text{additive}}}$$

$$\underline{\underline{F}} = \underline{\underline{I}} + \nabla \underline{u} = \underline{\underline{I}} + \mathrm{sym}(\nabla \underline{u}) + \mathrm{skew}(\nabla \underline{u})$$

$\Rightarrow$ nature of the rotation-stretch decomposition changes from multiplicative to additive!

Interpretation of components of $\underline{\underline{\varepsilon}}$

Start from $\underline{\underline{C}}$

$$\underline{\underline{\varepsilon}} = \underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}}) + O(\epsilon^2) \qquad |\nabla u| = \underset{\substack{\uparrow \\ \text{scalar}}}{\epsilon}$$

$$\Rightarrow \underline{\underline{C}} \approx \underline{\underline{I}} + 2\underline{\underline{\varepsilon}}$$

I) Diagonal components

$$C_{ii} = 1 + 2\varepsilon_{ii} \qquad \sqrt{C_{ii}} = \lambda(\underline{e}_i)$$

$$\lambda(\underline{e}_i) = \sqrt{1 + 2\varepsilon_{ii}}$$

Expand in Taylor-Series: $\sqrt{1-x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \dots$

where $x = 2\varepsilon_{ii}$ s.t. $\lambda(\underline{e}_i) = 1 + \varepsilon_{ii}$

$$\Rightarrow \quad \boxed{\varepsilon_{ii} \approx \lambda(\underline{e}_i) - 1} \quad \text{engineering strain in coordinate directions}$$

$$\lambda = \frac{|\underline{y} - \underline{x}|}{|\underline{Y} - \underline{X}|} = \frac{\ell}{L} \ \text{stretch}$$

$$\lambda - 1 = \frac{|\underline{y}-\underline{x}| - |\underline{Y}-\underline{X}|}{|\underline{Y}-\underline{X}|} = \frac{\ell - L}{L} = \frac{\Delta\ell}{L}$$

ε_{ii} = <u>relative change</u> in Length

II) Off-Diagonal Components

$$C_{ij} = \lambda(\underline{e}_i)\,\lambda(\underline{e}_j)\sin(\gamma_{ij}) \qquad \gamma_{ij} = \gamma(\underline{e}_i, \underline{e}_j)$$

$$C_{ij} \approx 2\,\varepsilon_{ij} \qquad i \neq j$$

$$\varepsilon_{ij} \approx \frac{1}{2}\lambda(\underline{e}_i)\,\lambda(\underline{e}_j)\,\sin(\gamma_{ij})$$

$$\lambda(\underline{e}_i) = 1 + \varepsilon_{ii} = 1 + O(\epsilon)$$

$$\lambda(e_j) = 1 + \varepsilon_{jj} = 1 + O(\epsilon)$$

$$\sin(\gamma_{ij}) = O(\epsilon)$$

$$\varepsilon_{ij} = \frac{1}{2}(1+\varepsilon_{ii})(1+\varepsilon_{jj})\,\sin(\gamma_{ij})$$

$$= \frac{1}{2}\Big[\sin(\gamma_{ij}) + \underbrace{\varepsilon_{ii}\sin(\gamma_{ij})}_{O(\epsilon^2)} + \underbrace{\varepsilon_{jj}\sin(\gamma_{ij})}_{O(\epsilon^2)} + \underbrace{\varepsilon_{ii}\,\varepsilon_{jj}\sin(\gamma_{ij})}_{O(\epsilon^3)}\Big]$$

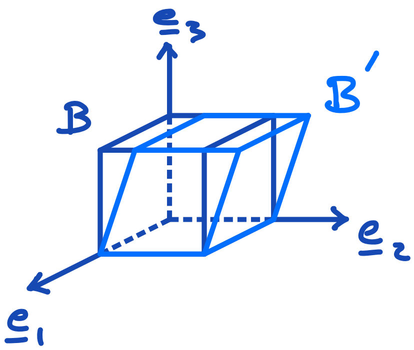
$$\Rightarrow \qquad \varepsilon_{ij} = \frac{1}{2}\sin(\gamma(\underline{e}_i, \underline{e}_j))$$

Infinitesimal strain $\Rightarrow$ $\gamma \ll 1$ $\Rightarrow$ $\sin\gamma \approx \gamma$

$$\boxed{\varepsilon_{ij} \approx \frac{1}{2}\gamma(\underline{e}_i, \underline{e}_j)}$$

$\Rightarrow$ half shear angle between coord. directions

Example: Simple shear



$$\underline{x} = \underline{\varphi}(\underline{X}) = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 + \alpha X_3 \\ X_3 \end{bmatrix} \qquad \alpha > 0$$

large def: $\underline{\underline{C}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & \alpha & 1+\alpha^2 \end{bmatrix}$

Infinitesimal:

$$\underline{u} = \underline{x} - \underline{X} = \begin{bmatrix} 0 \\ \alpha X_3 \\ 0 \end{bmatrix} \qquad \nabla \underline{u} = \underline{\underline{H}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \end{bmatrix}$$

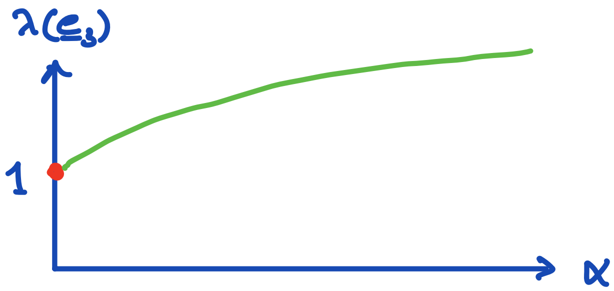
$$\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^T\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha/2 \\ 0 & \alpha/2 & 0 \end{bmatrix}$$

Infinitesimal streches: $\varepsilon_{ii} = \lambda(\underline{e}_i) - 1 = 0$

$\lambda(\underline{e}_i) = 1$ no stretch in any coord. direction

Finite streches: $C_{ii} = \lambda^2(\underline{e}_i)$ $\qquad C_{33} = \lambda^2(\underline{e}_3) = 1 + \alpha^2$

$\lambda(\underline{e}_3) = \sqrt{1+\alpha^2}$



– finite strain

– infinitesimal strain

Infinitesimal shear: $\varepsilon_{ij} \approx \frac{1}{2} \gamma(\underline{e}_i, \underline{e}_j)$

$$\gamma(\underline{e}_2, \underline{e}_3) = 2\,\varepsilon_{23} = \alpha$$

Finite shear: $\gamma(\underline{e}_2, \underline{e}_3) = \operatorname{asin}\left(\frac{\alpha}{\sqrt{1+\alpha^2}}\right)$