Example: $\nabla \cdot (\underline{\underline{S}}^T \underline{v})$ note $\underline{\underline{S}} = \underline{\underline{S}}(x)$ and $\underline{v} = v(\underline{x})$

$$q(x) = \underline{\underline{S}}^T(x)\, \underline{v}(\underline{x}) \qquad q_j = S_{ij} v_i$$

$$\nabla \cdot q = tr(q) = q_{j,j} = (S_{ij} v_i)_{,j}$$

$$= S_{ij,j} v_i + S_{ij} v_{i,j}$$

$$= (\nabla \cdot \underline{\underline{S}}) \cdot \underline{v} + \underline{\underline{S}} : \nabla \underline{v} \quad ✓$$

$\rightarrow$ useful for energy balance!

Example: $\nabla(\phi \underline{v}) = (\phi v_i)_{,j}\, \underline{e}_i \otimes \underline{e}_j$

$$= (\phi_{,j} v_i + \phi\, v_{i,j})\, \underline{e}_i \otimes \underline{e}_j$$

$$= v_i \phi_{,j}\, \underline{e}_i \otimes \underline{e}_j + \phi\, v_{i,j}\, \underline{e}_i \otimes \underline{e}_j$$

$$= \underline{v} \otimes \nabla \phi + \phi \nabla \underline{v} \quad ✓$$

Curl of a vector field

To any $\underline{v}(\underline{x}) \in \mathcal{V}$ we associate another vector field $\nabla \times \underline{v}$ defined by

$$\boxed{(\nabla \times \underline{v}) \times \underline{a} = (\nabla \underline{v} - \nabla \underline{v}^T)\, \underline{a}} \quad \text{for all } \underline{a} \in \mathcal{V}$$

Here $\underline{\omega} = \nabla \times \underline{v}$ is the axial vector of

$\underline{\underline{T}} = \nabla \underline{v} - \nabla \underline{v}^T = 2\,\mathrm{skew}(\nabla \underline{v})$

In index notation

$$w_j = \tfrac{1}{2}\epsilon_{ijk} T_{ik} = \tfrac{1}{2}\epsilon_{ijk}(v_{i,k} - v_{k,i})$$

$$= \tfrac{1}{2}(\epsilon_{ijk} v_{i,k} - \epsilon_{ijk} v_{k,i}) \qquad \epsilon_{ijk} = -\epsilon_{kji}$$

$$= \tfrac{1}{2}(\epsilon_{ijk} v_{i,k} + \epsilon_{kji} v_{k,i}) \qquad \text{flip } i \leftrightarrow k \text{ in second}$$

$$w_j = \epsilon_{ijk} v_{i,k}$$

$$\Rightarrow \quad \boxed{\underline{\omega} = \nabla \times \underline{v} = \epsilon_{ijk} v_{i,k} \underline{e}_j}$$

Note: Equivalently $\nabla \times \underline{v} = -\epsilon_{ijk} v_{i,j} \underline{e}_k$

by switching & renaming indices

Explicitly: $\nabla \times \underline{v} = (v_{3,2} - v_{2,3})\underline{e}_1 + (v_{1,3} - v_{3,1})\underline{e}_2$

$$+ (v_{2,1} - v_{1,2})\underline{e}_3$$

Physical interpretation:

If $\underline{v}$ is a velocity field then $\nabla \times \underline{v}$

measures the angular velocity.

If $\nabla \times \underline{v} = 0 \Rightarrow \underline{v}(x)$ is irrotational/conservative

Further we can show

$\boxed{\nabla \times \nabla \phi = \underline{0}}$ and $\boxed{\nabla \cdot (\nabla \times \underline{v}) = 0}$ $\Rightarrow$ HW3

this follows as

$$\nabla \times \nabla \phi = \nabla \times (\phi_{,i} \underline{e}_i) = \epsilon_{ijk} (\phi_{,i})_{,k} \underline{e}_j$$

$$= \epsilon_{ijk} \phi_{,ik} \underline{e}_j$$

$$= \tfrac{1}{2} (\epsilon_{ijk} \phi_{,ik} + \epsilon_{ijk} \phi_{,ik}) \underline{e}_j$$

2^{nd} term $\epsilon_{ijk} = -\epsilon_{kji}$

$$= \tfrac{1}{2} (\epsilon_{ijk} \phi_{,ik} - \epsilon_{kji} \phi_{,ik}) \underline{e}_j$$

$\phi_{,ik} = \phi_{,ki}$

$$= \tfrac{1}{2} (\epsilon_{ijk} \phi_{,ik} - \epsilon_{kji} \phi_{,ki}) \underline{e}_j$$

rename dummys in second term $i \leftrightarrow j$

$$= \tfrac{1}{2} (\epsilon_{ijk} \phi_{,ik} - \epsilon_{ijk} \phi_{,ik}) \underline{e}_j$$

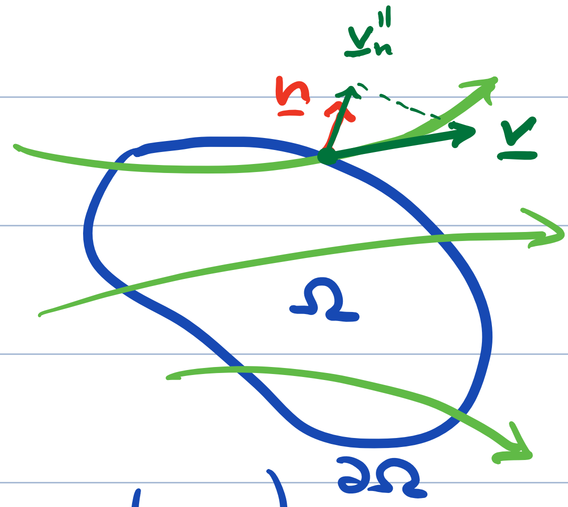
$$= \underline{0}$$

Integral theorems

Essential to derive balance laws

Vector divergence theorem

For any $\underline{v}(\underline{x}) \in \mathcal{V}$ we have

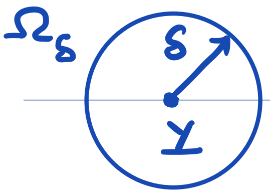


$$\boxed{\begin{aligned} \int_{\partial\Omega} \underline{v}\cdot\underline{n}\, dA &= \int_{\Omega} \nabla\cdot\underline{v}\, dV \\ \int_{\partial\Omega} v_i\, n_i\, dA &= \int_{\Omega} v_{i,i}\, dV \end{aligned}}$$

(for proof see vector calculus class)

Physical Interpretation:

Here $\underline{v}$ is either a velocity $\left[\frac{L}{T}\right]$ or a volumetric flux $\left[\frac{L^3}{L^2 T} = \frac{L}{T}\right]$. The units of $\int_{\partial\Omega} \underline{v}\cdot\underline{n}\, dA$ are then $\left[\frac{L^3}{T}\right]$ so that the l.h.s. represents the rate at which volume is leaving or entering Ω.



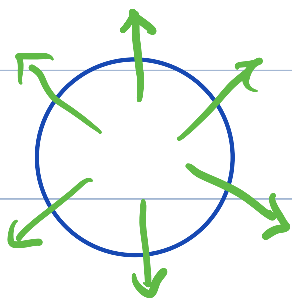
$$\int_{\partial\Omega_\delta} \underline{v}\cdot\underline{n}\, dA = \int_{\Omega_\delta} \nabla\cdot\underline{v}\, dV$$

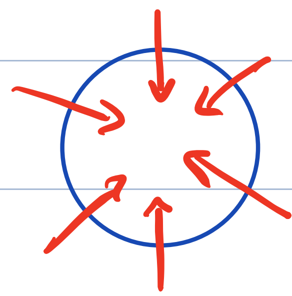
$$\lim_{\delta\to 0} \int_{\Omega_\delta} \nabla\cdot\underline{v}\, dV = V_\delta\, \nabla\cdot\underline{v}\big|_{\underline{x}}$$ V_δ = vol. of sphere

$$\boxed{\nabla\cdot\underline{v}\big|_{\underline{x}} = \lim_{\delta\to 0} \frac{1}{V_\delta} \int_{\partial\Omega} \underline{v}\cdot\underline{n}\, dA}$$

Divergence is the point wise rate of volume expansion/contraction.

$\nabla\cdot\underline{v} > 0$





$\nabla\cdot\underline{v} < 0$

Incompressible flows/deformations are solenoidal $\nabla\cdot\underline{v} = 0$.

Tensor divergence theorem

For any $\underline{\underline{S}}(\underline{x}) \in \mathcal{V}^2$ on domain Ω with boundary $\partial\Omega$ we have

$$\boxed{\begin{aligned} \int_{\partial\Omega} \underline{\underline{S}}\,\underline{n}\, dA &= \int_{\Omega} \nabla \cdot \underline{\underline{S}}\, dV \\ \int_{\partial\Omega} S_{ij} n_j\, dA &= \int_{\Omega} S_{ij,j}\, dV \end{aligned}}$$

To derive this from vector divergence Thm consider arbitrary constant vector $\underline{a} \in \mathcal{V}$

$$\underline{a} \cdot \int_{\partial\Omega} \underline{\underline{S}}\,\underline{n}\, dA = \int_{\partial\Omega} \underline{a} \cdot \underline{\underline{S}}\,\underline{n}\, dA = \int_{\partial\Omega} (\underline{\underline{S}}^T \underline{a}) \cdot \underline{n}\, dA$$

where $\underline{\underline{S}}^T \underline{a}$ is a vector and we can apply vector divergence Thm

$$\int_{\partial\Omega} (\underline{\underline{S}}^T \underline{a}) \cdot \hat{\underline{n}}\, dA = \int_{\Omega} \nabla \cdot (\underline{\underline{S}}^T \underline{a})\, dV$$

using the definition: $(\nabla \cdot \underline{\underline{S}}) \cdot \underline{a} = \nabla \cdot (\underline{\underline{S}}^T \underline{a})$

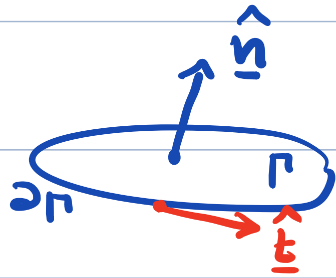
$$\int_{\partial\Omega} (\underline{\underline{S}}^T \underline{a}) \cdot \hat{\underline{n}}\, dA = \int_{\Omega} (\nabla \cdot \underline{\underline{S}}) \cdot \underline{a}\, dV$$

using def. of transpose and that $\underline{a}$ is const.

$$\underline{a} \cdot \int_{\partial\Omega} \underline{\underline{S}}\,\hat{\underline{n}}\, dA = \underline{a} \cdot \int_{\Omega} \nabla \cdot \underline{\underline{S}}\, dV$$

the result follows from arbitrariness of $\underline{a}$

Stokes Thm

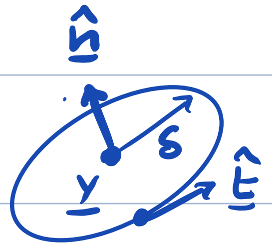


Consider surface Γ with boundary $\partial\Gamma$, unit normal $\hat{\underline{n}}$ and unit tangent (right-handed). Then for any $\underline{v}(\underline{x}) \in \mathcal{V}$ we have

$$\boxed{\int_{\Gamma} (\nabla \times \underline{v}) \cdot \hat{\underline{n}}\, dA = \oint_{\partial\Gamma} \underline{v} \cdot \hat{\underline{t}}\, ds}$$

Here $\oint_{\partial\Gamma} \underline{v} \cdot \hat{\underline{t}}\, ds$ is the circulation of $\underline{v}$ around $\partial\Gamma$.

Physical Interpretation:



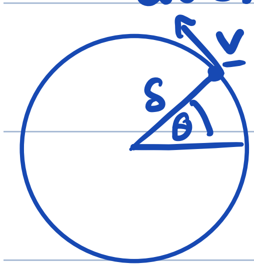
Γ_δ is a disk of radius δ around $\underline{y}$.

$$\oint_{\partial\Gamma} \underline{v}(\underline{x}) \cdot \hat{\underline{t}}(\underline{x})\, ds = \int_\Gamma (\nabla \times \underline{v})(\underline{x}) \cdot \hat{\underline{n}}\, dA$$

In the limit of $\delta \to 0$

$$\underbrace{\overline{\underline{v} \cdot \hat{\underline{t}}}\Big|_{\underline{y}}}_{\text{ave. tangential velocity}} 2\pi\delta \approx \nabla \times \underline{v}\Big|_{\underline{y}} \cdot \hat{\underline{n}}\ \pi\delta^2$$

ave. tangential velocity ~ angular velocity



angular velocity : $\omega = \dfrac{d\theta}{dt}$

$$|\underline{v}| = \omega\delta$$

$$\Rightarrow \quad \overline{\underline{v} \cdot \hat{\underline{t}}}\Big|_{\underline{y}} = \omega\delta$$

$$2\pi\delta^2\omega \approx \nabla \times \underline{v}\Big|_{\underline{y}} \cdot \hat{\underline{n}}\ \pi\delta^2$$

$$2\omega = \nabla \times \underline{v}\Big|_{\underline{y}} \cdot \hat{\underline{n}} \qquad\qquad \hat{\underline{n}} = \frac{\nabla \times \underline{v}}{|\nabla \times \underline{v}|}\Bigg|_{\underline{y}}$$

$$2\omega = \frac{(\nabla \times \underline{v}|_{\underline{y}}) \cdot (\nabla \times \underline{v}|_{\underline{y}})}{|\nabla \times \underline{v}|_{\underline{y}}} = |\nabla \times \underline{v}|_{\underline{y}}$$

$$\Rightarrow \quad \boxed{|\nabla \times \underline{v}|_{\underline{y}} = 2\omega}$$

Curl of $\underline{v}$ is twice the angular velocity.

Derivatives of tensor functions

So far we have considered fields: $\phi(\underline{x}), \underline{v}(\underline{x}), \underline{\underline{S}}(\underline{x})$

Now we are interested in tensor functions

- scalar-valued tensor functions: $\psi = \psi(\underline{\underline{S}})$
- tensor-valued tensor functions: $\underline{\underline{\Sigma}} = \underline{\underline{\Sigma}}(\underline{\underline{S}})$

Derivatives of scalar-valued tensor functions

Typical examples: $\det \underline{\underline{A}}$ or $\operatorname{tr} \underline{\underline{A}}$

Definition: A function $\psi(\underline{\underline{S}})$ is differentiable at $\underline{\underline{A}}$ if there exists a tensor $D\psi(\underline{\underline{A}})$, s.t.

$$\psi(\underline{\underline{A}} + \underline{\underline{H}}) = \psi(\underline{\underline{A}}) + D\psi(\underline{\underline{A}}) : \underline{\underline{H}} + o(|\underline{\underline{H}}|)$$

or equivalently with $\underline{\underline{H}} = \epsilon \underline{\underline{U}}$

$$\boxed{D\psi(\underline{\underline{A}}) : \underline{\underline{U}} = \frac{d}{d\epsilon} \psi(\underline{\underline{A}} + \epsilon \underline{\underline{U}})\Big|_{\epsilon=0}} \qquad \text{for all } \underline{\underline{U}} \in V^2$$

$D\psi(\underline{\underline{A}})$ is called the derivative of ψ at $\underline{\underline{A}}$

In frame $\{\underline{e}_i\}$ we have $\boxed{D\psi(\underline{\underline{A}}) = \frac{\partial \psi}{\partial A_{ij}} \underline{e}_i \otimes \underline{e}_j}$

To see this write $\psi(A_{11}, A_{12}, \dots, A_{33})$

and $\underline{\underline{U}} = U_{kl}\, \underline{e}_k \otimes \underline{e}_l$ then

$$\psi(\underline{\underline{\bar{A}}} + \epsilon \underline{\underline{U}}) = \psi(\underbrace{\bar{A}_{11} + \epsilon U_{11}}_{A_{11}}, \underbrace{\bar{A}_{12} + \epsilon U_{12}}_{A_{12}}, \dots, \bar{A}_{33} + \epsilon U_{33})$$

by chain rule

$$D\psi(\underline{\underline{A}}) : \underline{\underline{U}} = \frac{d}{d\varepsilon} \psi(\bar{A}_{11} + \epsilon U_{11}, \dots, \bar{A}_{33} + \epsilon U_{33}) \Big|_{\epsilon = 0}$$

$$= \frac{\partial \psi}{\partial A_{11}} U_{11} + \frac{\partial \psi}{\partial A_{12}} U_{12} + \dots + \frac{\partial \psi}{\partial A_{33}} U_{33} = \frac{\partial \psi}{\partial A_{ij}} U_{ij}$$

$$= \left(\frac{\partial \psi}{\partial A_{ij}} \underline{e}_i \otimes \underline{e}_j \right) : \left(U_{kl}\, \underline{e}_k \otimes \underline{e}_l \right)$$

result is implied by the arbitraryness of $\underline{\underline{U}}$

Derivative of trace

$\psi(\underline{\underline{A}}) = \mathrm{tr}(\underline{\underline{A}}) = A_{ii}$ Using the definition

$$D\mathrm{tr}(\underline{\underline{A}}) = \frac{\partial A_{ii}}{\partial A_{kl}} \underline{e}_k \otimes \underline{e}_l = \delta_{ik} \delta_{il}\, \underline{e}_k \otimes \underline{e}_l = \underline{e}_i \otimes \underline{e}_i = \underline{\underline{I}}$$

$$\boxed{D\mathrm{tr}(\underline{\underline{A}}) = \underline{\underline{I}}}$$

Derivative of determinant

Let $\psi(\underline{\underline{A}}) = \det(\underline{\underline{A}})$, if $\underline{\underline{A}}$ is invertible

$$\boxed{D\det(\underline{\underline{A}}) = \det(\underline{\underline{A}})\,\underline{\underline{A}}^{-T}}$$

Note this takes some work !

Start by using the directional derivative

$$D\det(\underline{\underline{A}} + \epsilon\underline{\underline{U}}) = \frac{d}{d\epsilon}\det(\underline{\underline{A}} + \epsilon\underline{\underline{U}})\Big|_{\epsilon=0}$$

First simplify expansion

$$\det(\epsilon\underline{\underline{U}} + \underline{\underline{A}}) = \det\left(\epsilon\underline{\underline{A}}\left(\underline{\underline{A}}^{-1}\underline{\underline{U}} + \frac{1}{\epsilon}\underline{\underline{I}}\right)\right) \qquad \frac{1}{\epsilon} = -\lambda$$

$$= \det(\epsilon\underline{\underline{A}})\det(\underline{\underline{A}}^{-1}\underline{\underline{U}} - \lambda\underline{\underline{I}})$$

$$= \epsilon^3\det(\underline{\underline{A}})\det(\underline{\underline{A}}^{-1}\underline{\underline{U}} - \lambda\underline{\underline{I}})$$

from definition of principal invariants

$$\det(\underline{\underline{A}}^{-1}\underline{\underline{U}} - \lambda\underline{\underline{I}}) = -\lambda^3 + \lambda^2 I_1(\underline{\underline{A}}^{-1}\underline{\underline{U}}) - \lambda I_2(\underline{\underline{A}}^{-1}\underline{\underline{B}}) + I_3(\underline{\underline{A}}^{-1}\underline{\underline{B}})$$

$$= -\left(-\frac{1}{\epsilon}\right)^3 + \left(-\frac{1}{\epsilon}\right)^2 I_1 + \frac{1}{\epsilon}I_2 + I_3$$

$$= \frac{1}{\epsilon^3} + \frac{1}{\epsilon^2}I_1 + \frac{1}{\epsilon}I_2 + I_3$$

substituting above $\Rightarrow$ expansion in ϵ

$$\det(\underline{\underline{A}}+\epsilon\underline{\underline{U}}) = \epsilon^3 \det(\underline{\underline{A}})\left(\frac{1}{\epsilon^3}+\frac{1}{\epsilon^2}I_1+\frac{1}{\epsilon}I_2+I_3\right)$$

$$= \det(\underline{\underline{A}})\,(1+\epsilon I_1+\epsilon^2 I_2+\epsilon^3 I_3)$$

substitute into directional derivative

$$D\det(\underline{\underline{A}}):\underline{\underline{U}} = \frac{d}{d\epsilon}\det(\underline{\underline{A}}+\epsilon\underline{\underline{U}})\Big|_{\epsilon=0} = \det(\underline{\underline{A}})\,I_1(\underline{\underline{A}}^{-1}\underline{\underline{U}})$$

since $I_1(\underline{\underline{A}}^{-1}\underline{\underline{U}}) = \mathrm{tr}(\underline{\underline{A}}^{-1}\underline{\underline{U}}) = \mathrm{tr}(A^{-1}_{ij}U_{jk}\underline{e}_i\otimes\underline{e}_k)$

$$= A^{-1}_{ij}U_{ji} = A^{-T}_{ji}U_{ji} = \underline{\underline{A}}^{-T}:\underline{\underline{U}}$$

so that

$$\underline{D\det(\underline{\underline{A}})}:\underline{\underline{U}} = \underline{\det(\underline{\underline{A}})\,\underline{\underline{A}}^{-T}}:\underline{\underline{U}}$$

the result follows from the arbitraryness of $\underline{\underline{U}}$

Time derivative of scalar valued tensor function

Let $\underline{\underline{S}} = \underline{\underline{S}}(t) \in \mathcal{V}^2$. In stationary frame $\{\underline{e}_i\}$

$$\underline{\underline{S}}(t) = S_{ij}(t)\, \underline{e}_i \otimes \underline{e}_j \quad \text{so that}$$

$$\boxed{\dot{\underline{\underline{S}}} = \frac{d\underline{\underline{S}}}{dt} = \frac{dS_{ij}}{dt}\, \underline{e}_i \otimes \underline{e}_j}$$

How do we compute $\frac{d}{dt} \psi(\underline{\underline{S}}(t))$?

By the chain rule we have

$$\begin{aligned}
\frac{d}{dt} \psi(\underline{\underline{S}}(t)) &= \frac{d}{dt} \psi(S_{11}(t), S_{12}(t), \ldots, S_{33}(t)) \\
&= \frac{\partial \psi}{\partial S_{11}} \frac{dS_{11}}{dt} + \ldots + \frac{\partial \psi}{\partial S_{33}} \frac{dS_{33}}{dt} = \frac{\partial \psi}{\partial S_{ij}} \frac{dS_{ij}}{dt} \\
&= D\psi(\underline{\underline{S}}) : \dot{\underline{\underline{S}}}
\end{aligned}$$

$\Rightarrow$ chain rule leads to a contraction

$$\boxed{\frac{d}{dt} \psi(\underline{\underline{S}}(t)) = D\psi(\underline{\underline{S}}) : \dot{\underline{\underline{S}}}}$$

Example: $\boxed{\frac{d}{dt} \det(\underline{\underline{S}}(t)) = \det(\underline{\underline{S}})\, \underline{\underline{S}}^{-T} : \dot{\underline{\underline{S}}}}$ Jacobi's formula