Newtonian Fluids

A fluid is incompressible Newtonian if:

1) Reference mass density uniform: $\rho_0(X) = \rho_0$

2) Fluid is incompressible $\nabla_x \cdot \underline{v} = 0$

Prop 1 + 2 $\Rightarrow$ $\rho(\underline{x}, t) = \rho_0 > 0$

3) Cauchy stress field is Newtonian

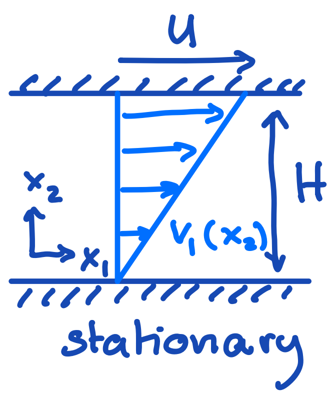
$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^r + \underline{\underline{\sigma}}^a$$

Reactive stress: $\underline{\underline{\sigma}}^r = -p\,\underline{\underline{I}}$

p is multiplier for $\nabla_x \cdot \underline{v} = 0$

same as Ideal fluid

Active stress: $\Rightarrow$ Newton's law of viscosity

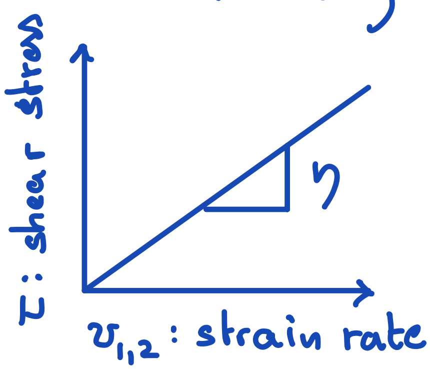


Shear force on top plate

$$F = \eta\, A\, \frac{U}{H} \qquad A = \text{area of plate}$$

η = dynamic/absolute viscosity

units $\left[\frac{M}{LT}\right]$ e.g. Pa s



shear stress: $\tau = \frac{F}{A}$

$$\boxed{\tau = \eta\, \frac{\partial v_1}{\partial x_2}}$$ Newton's law of viscosity

Generalization to 3D

$$\sigma^a_{ij} = \mu \frac{\partial v_i}{\partial x_j} \Rightarrow \underline{\underline{\sigma}}^a = \mu \nabla \underline{\underline{\sigma}}$$

but $\nabla \underline{v}$ <u>not</u> objective $\Rightarrow \underline{\underline{\sigma}}^a = 2\mu \, \mathrm{sym}(\nabla_x \underline{v})$

$\Rightarrow$ tensor version of classical linear laws

Fourier's law:	$\underline{q} = -\kappa \nabla T$	(heat conduction)
Fick's law:	$\underline{j} = -D \nabla c$	(species diffusion)
Newton's law:	$\underline{\underline{\sigma}} = \eta \, \mathrm{sym}(\nabla \underline{v})$	(momentum diffusion)

Cauchy stress in Newtonian fluid:

$$\boxed{\underline{\underline{\sigma}} = -p \underline{\underline{I}} + \eta (\nabla \underline{v} + \nabla \underline{v})}$$

In limit $\eta \to 0$ Newtonian fluid reduces to ideal fluid.

Compare to Representation Thm with $A = \nabla \underline{v}$:

$$\underline{\underline{G}}(\underline{\underline{A}}) = \lambda \, \mathrm{tr}(\underline{\underline{A}}) \underline{\underline{I}} + 2\mu \, \mathrm{sym}(\underline{\underline{A}})$$

$$\underline{\underline{G}}(\underline{\underline{A}}) = \lambda \cancel{(\nabla \cdot \underline{v})} \underline{\underline{I}} + \mu (\nabla \underline{v} + \nabla \underline{v}^T) = \underline{\underline{\sigma}}^a$$

where $\mu = \eta$

Navier - Stokes Equations

Setting $\rho = \rho_0$ and $\underline{\underline{\sigma}} = -p\underline{\underline{I}} + \eta(\nabla\underline{v} + \nabla\underline{v}^T)$

we obtain lin. mom. balance

$$\rho\dot{\underline{v}} = \nabla\cdot\left(-p\underline{\underline{I}} + \eta(\nabla\underline{v} + \nabla\underline{v}^T)\right) + \rho\underline{g}$$

assuming $\eta = \text{constant}$ we have

$$\nabla\cdot\underline{\underline{\sigma}} = -\nabla p + \eta\nabla^2\underline{v} + \eta\nabla\cdot(\nabla\underline{v})^T$$

Last term

$$\nabla\cdot(\nabla\underline{v})^T = v_{j,ij}\,\underline{e}_i = v_{j,ji}\,\underline{e}_i = \nabla(\cancelto{0}{\nabla\cdot\underline{v}})$$

$$\Rightarrow \nabla\cdot\underline{\underline{\sigma}} = -\nabla p + \eta\nabla^2\underline{v}$$

expanding mat. deriv.: $\dot{\underline{v}} = \frac{\partial\underline{v}}{\partial t} + (\nabla\underline{v})\underline{v}$

$$\boxed{\begin{gathered}\rho\left[\frac{\partial\underline{v}}{\partial t} + (\nabla\underline{v})\underline{v}\right] = \mu\nabla^2\underline{v} - \nabla p + \rho\underline{g}\\ \nabla\cdot\underline{v} = 0\end{gathered}}$$

Convective form of Navier-Stokes Equation

Scaling Navier Stokes Equations

$$\rho \frac{\partial \underline{v}}{\partial t} + (\nabla \underline{v})\, \underline{v} = \eta \nabla^2 \underline{v} - \nabla p + \rho g$$

reduced pressure: $\pi = p + \rho g z$

$$-\nabla p + \rho g = -\nabla p - \rho g \hat{z} = -\nabla (p + \rho g z) = -\nabla \pi$$

we have

$$\rho \left(\frac{\partial \underline{v}}{\partial t} + (\nabla \underline{v})\, \underline{v} \right) - \eta \nabla^2 \underline{v} = -\nabla \pi$$

Non-dimensionalize with generic quantities to define standard dimensionless parameters.

- Dependent variables: $\underline{v}$, π
- Independent variables: $\underline{x}$, t
- Parameters: $\rho \left[\frac{M}{L^3}\right]$ $\mu \left[\frac{M}{LT}\right] \rightarrow \nu = \frac{\mu}{\rho} \left[\frac{L^2}{T}\right]$

  + Geometry, BC, IC

Use parameters to scale the variables:

$$\underline{v}' = \frac{\underline{v}}{v_c} \qquad \pi' = \frac{\pi}{\pi_c} \qquad \underline{x}' = \frac{\underline{x}}{x_c} \qquad t' = \frac{t}{t_c}$$

Note: $\underline{v}'$, π', $\underline{x}'$ and t' are dimensionless substitute into governing equations

$$\rho \frac{v_c}{t_c} \frac{\partial \underline{v}'}{\partial t'} + \frac{\rho v_c^2}{x_c} (\nabla_x' \underline{v}') \underline{v}' - \frac{\mu v_c}{x_c^2} \nabla_x'^2 \underline{v}' = -\frac{\pi_c}{x_c} \nabla_x' \pi'$$

Choose to scale to accumulation term

$$\frac{\partial \underline{v}'}{\partial t'} + \underbrace{\frac{v_c t_c}{x_c}}_{\Pi_1} (\nabla_x' \underline{v}') \underline{v}' - \underbrace{\frac{\nu t_c}{x_c^2}}_{\Pi_2} \nabla_x'^2 \underline{v}' = -\underbrace{\frac{\pi_c t_c}{x_c \rho_o v_c}}_{\Pi_3} \nabla_x' \pi'$$

where $\nu = \frac{\eta}{\rho}$ kinematic viscosity

$\nu \; [\frac{L^2}{T}]$ momentum diffusivity

Three dimensionless groups $\Rightarrow$ define time scale

$\Pi_1 = \frac{v_c t_c}{x_c} = 1 \Rightarrow$ advective scale $\quad t_c = t_A = \frac{x_c}{v_c}$

$\Pi_2 = \frac{\nu t_c}{x_c^2} = 1 \Rightarrow$ diffusive scale $\quad t_c = t_D = \frac{x_c^2}{\nu}$

Use Π_3 to define pressure scale

$$\Pi_3 = \frac{\pi_c t_c}{x_c \rho_o v_c} = 1 \Rightarrow \pi_c = \frac{x_c \rho_o v_c}{t_c}.$$

Choose a diffusive time scale $t_c = \frac{x_c^2}{\nu}$

Dimensionless N-S equation:

$$\frac{\partial \underline{v}}{\partial t} + \frac{v_c x_c}{\nu}(\nabla'_x \underline{v}')\underline{v}' - \nabla'^2_x \underline{v} = -\nabla'_x \pi'$$

$\Rightarrow$ one remaining dim. less group

$$\boxed{Re = \frac{v_c x_c}{\nu}}$$ Reynolds number

ratio of time scales: $Re = \frac{t_D}{t_A} = \frac{\text{diffusive}}{\text{advective}}$

Hence we have (dropping primes)

$$\boxed{\frac{\partial \underline{v}}{\partial t} + Re\,(\nabla \underline{v})\underline{v} = \nabla^2 \underline{v} - \nabla \pi}$$

$\Rightarrow$ only one governing dimensionless parameter: Reynolds number

For viscous flow of glacier:

$\rho = 10^3 \frac{kg}{m^3}$ $\quad v_c = 100 \frac{m}{yr} \sim 10^{-6} \frac{m}{s}$

$\mu = 10^{14}$ Pas $\quad x_c = 10^2$ m (thickness)

$$Re = \frac{v_c x_c \rho_0}{\mu_c} = \frac{10^{-6+2+3}}{10^{14}} = 10^{-1-14} = 10^{-15} \ll 1$$

$\Rightarrow$ advective momentum transport is negligible

For $Re = 0$ we have

$$\boxed{\begin{gathered}\frac{\partial \underline{v}}{\partial t} - \nabla^2 \underline{v} = -\nabla \pi \\ \nabla \cdot \underline{v} = 0\end{gathered}}$$

Equations are linear !

But is it worth resolving diffusive timescale ?

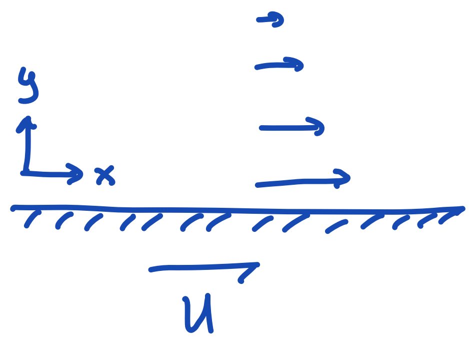
$$t_D = \frac{x_c^2 \rho_0}{\mu} = 10^{4+3-14}\,s = 10^{-7}\,s$$

This is very short compared to 100 years of glacier response. Not worth resolving transients.

Can't eliminate transient term because we scaled to it $\Rightarrow$ scale to diffusion term.

Rayleigh's problem

- Semi-infinite half-space
- Stationary fluid
- Impulsively started plate with velocity U.



$$v_c = U \rightarrow Re = \frac{U x_c \rho_0}{\mu} \ll 1 \Rightarrow U \ll \frac{\mu}{x_c \rho_0}$$

But what is x_c? Not obvious

Redimensionalize assuming $Re \ll 1$

$$\frac{\partial \underline{v}}{\partial t} - \nu \nabla^2 \underline{v} = -\nabla \pi \quad \& \quad \nabla \cdot \underline{v} = 0 \qquad \underline{v} = \begin{pmatrix} u \\ w \end{pmatrix}$$

Simplify the equations:

Domain is infinite in x but $|\pi| < \infty \Rightarrow \frac{\partial \pi}{\partial x} = 0$

Flow is horizontal: $\underline{v} = \begin{pmatrix} u \\ w \end{pmatrix} \Rightarrow w = 0$

From continuity: $\frac{\partial u}{\partial x} + \cancelto{0}{\frac{\partial w}{\partial y}} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0 \Rightarrow u = u(y)$

$$\nabla^2 \underline{v} = v_{i,jj} \underline{e}_i \qquad i,j \in \{1,2\}$$

$$= \begin{pmatrix} v_{1,11} & v_{1,22} \\ v_{2,11} & v_{2,22} \end{pmatrix} = \begin{pmatrix} \cancelto{0}{u_{xx}} & u_{yy} \\ \cancelto{0}{w_{xx}} & \cancelto{0}{w_{yy}} \end{pmatrix} = \begin{pmatrix} u_{yy} \\ 0 \end{pmatrix}$$

Substituting we have

x-mom.: $\frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial x^2} = 0$

y-mom.: $0 = -\frac{\partial \pi}{\partial y}$

$\Rightarrow \boxed{\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}}$ with $u(0,y) = 0$ $u(t,0) = U$

This is identical to heating a semi-infinite rod from the end.

Problem has self-similar solution in

$\eta = \frac{y}{\sqrt{4\nu t}}$ and $u(y,t) = U f(\eta)$

where $\sqrt{4\nu t}$ takes role of char. length that depends on t.

derivatives: $\frac{\partial \eta}{\partial t} = -\frac{1}{2}\frac{\eta}{t}$ $\frac{\partial \eta}{\partial y} = \frac{1}{\sqrt{4\nu t}}$

The derivatives of u transform as:

$\frac{\partial u}{\partial t} = U \frac{df}{d\eta}\frac{\partial \eta}{\partial t} = -\frac{U}{2}\frac{\eta}{t}$ and $\frac{\partial^2 u}{\partial x^2} = U \frac{d^2 f}{d\eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 = \frac{U}{4\nu t}\frac{d^2 f}{d\eta^2}$

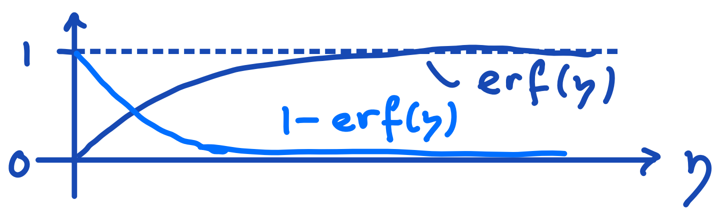
substituting into PDE:

$\frac{d^2 f}{d\eta^2} + 2\eta \frac{df}{d\eta} = 0$ with $f(\eta=0) = 1$

Reduce PDE in y and t to ODE in η

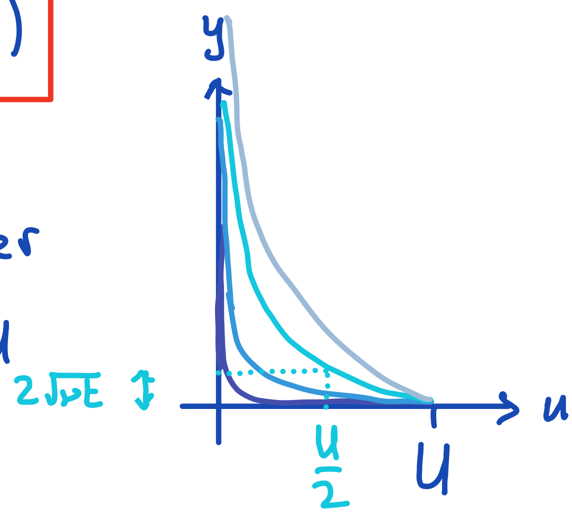
Solution : $f(\eta) = 1 - \mathrm{erf}(\eta)$ (Gauss)

where $\mathrm{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\xi^2} d\xi$ error function



Resubstituting for $f = \frac{u}{U}$ and $\eta = \frac{y}{\sqrt{4\nu t}}$

$$\boxed{u(y,t) = U\left(1 - \mathrm{erf}\left(\frac{y}{\sqrt{4Dt}}\right)\right)}$$



Diffusive boundary layer where momentum added by boundary penetrates into the quiescent fluid.

$\nu = \frac{\mu}{\rho_0}$ is Diffusion coefficient.

Mechanical energy considerations

Stress power of Newtonian fluid is

$$\underline{\underline{\sigma}} : \underline{\underline{d}} = (-p\underline{\underline{I}} + 2\mu \underline{\underline{d}}) : \underline{\underline{d}} = -p \underbrace{\underline{\underline{I}} : \underline{\underline{d}}}_{\nabla_x \cdot \underline{v} = 0} + 2\mu \, \underline{\underline{d}} : \underline{\underline{d}}$$

$$= 2\mu \, \underline{\underline{d}} : \underline{\underline{d}}$$

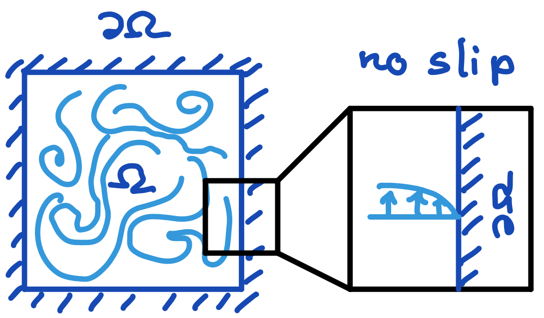
From reduced Clausius-Duhem inequality

$$\rho \dot{\psi} \leq 2\mu \underbrace{\underline{\underline{d}} : \underline{\underline{d}}}_{\geq 0}$$

$\Rightarrow$ only if $\mu > 0$ energy is dissipated during the flow $\dot{\psi} < 0$

Kinetic Energy of Fluid Motion

Dissipation of kinetic energy in a closed domain with no-slip boundaries.



$\Rightarrow$ Energy is lost by boundary

Change in kinetic energy

$$\frac{d}{dt} K(t) = \int_\Omega \frac{1}{2}\rho \frac{d}{dt} |\underline{v}|^2 \, dV = \int_\Omega \rho \, \underline{\dot{v}} \cdot \underline{v} \, dV$$

from Navier-Stokes Eqns: $\rho_0 \dot{\underline{v}} = \mu \nabla_x^2 \underline{v} - \nabla \underline{\Psi}$

$$\frac{d}{dt} K(t) = \int_\Omega \left(\mu \nabla_x^2 \underline{v} - \nabla \underline{\Psi}\right) \cdot \underline{v} \, dV_x$$

1) Integration by parts in fixed domain Ω with "no slip" boundaries $\underline{v}=\underline{0}$ on $\partial\Omega$.

$$\boxed{\int_\Omega (\nabla_x^2 \underline{v})\cdot\underline{v}\, dV_x = -\int_\Omega (\nabla_x \underline{v}):(\nabla_x \underline{v})\, dV_x}$$

To see this consider $(v_{i,j}\, v_i)_{,j} = v_{i,jj}\, v_i + v_{i,j}\, v_{i,j}$

$$(\nabla_x^2 \underline{v})\cdot\underline{v} = v_{i,jj}\, v_i = (v_{i,j}\, v_i)_{,j} - v_{i,j}\, v_{i,j}$$
$$= \nabla\cdot((\nabla_x \underline{v})^T \underline{v}) - (\nabla_x \underline{v}):(\nabla_x \underline{v})$$

substituting into integral and applying div-thm

$$\int_\Omega (\nabla_x^2 \underline{v})\cdot\underline{v}\, dV_x = \int_{\partial\Omega} (\nabla_x \underline{v})^T \cancelto{0}{\underline{v}}\cdot\underline{n}\, dA_x - \int_\Omega (\nabla_x \underline{v}):(\nabla_x \underline{v})\, dV_x$$

2) Poincaré Inequality

$$\|\underline{u}\|_\Omega \le \lambda \|\nabla \underline{u}\|_\Omega \quad \text{for } \underline{u}=0 \quad \partial\Omega \quad \lambda>0$$

using standard inner product

$$\boxed{\int_\Omega |\underline{u}|^2\, dV_x \le \lambda \int_\Omega \nabla_x \underline{u} : \nabla_x \underline{u}\, dV_x}$$

Notice λ has units of L^2 and scales with area of Ω.

Kinetic Energy of Newtonian & Ideal fluids

Consider a fixed domain Ω with $\underline{v}=0$ on $\partial\Omega$

and a conservative body force. $b = -\nabla_x \Phi$.

The kinetic energy is given by

$$K(t) = \int_{\Omega} \frac{1}{2} \rho_0 |\underline{v}|^2 \, dV_x \quad \text{and} \quad K(0) = K_0$$

I) Newtonian fluid

$$\boxed{K(t) \leq e^{-2\mu t/\lambda \rho_0} K_0}$$

The kinetic energy of a Newtonian fluid dissipates to zero exponentially fast.

II) Ideal fluid

$$\boxed{K(t) = K_0}$$

The kinetic energy of ideal fluid is constant.

show $\int_\Omega \nabla_x \psi \cdot \underline{v} \, dV_x = 0$

$$\nabla_x \cdot (\psi \underline{v}) = \nabla_x \psi \cdot \underline{v} + \cancelto{0}{(\nabla_x \cdot \underline{v})} \psi = \nabla_x \psi \cdot \underline{v}$$

substitute and use Div-Thm

$$\frac{d}{dt} K(t) = \int_\Omega \mu (\nabla_x^2 \underline{v}) \cdot \underline{v} \, dV_x - \int_{\partial\Omega} \psi \cancelto{0}{\underline{v}} \cdot \underline{n} \, dA_x$$

using integration by parts

$$\frac{d}{dt} K(t) = -\mu \int_\Omega (\nabla_x \underline{v}) : (\nabla_x \underline{v}) \, dV_x$$

for Ideal fluid $\mu = 0 \Rightarrow K(t) = K_0$

for Newtonian fluid apply Pointcare inequality

$$\frac{d}{dt} K(t) \leq -\frac{\mu}{\lambda} \int_\Omega |\underline{v}|^2 dV_x = -\frac{2\mu}{\lambda \rho_0} K(t)$$

so that we have

$$\boxed{\frac{d}{dt} K(t) \leq -\frac{2\mu}{\lambda \rho_0} K(t)}$$

where λ depends on area of the domain.

Solve by seperation of parts

$$\frac{dK}{K} \leq -\frac{2\mu}{\rho_0 \lambda}\, dt = -\alpha\, dt$$

$$\ln k \leq -\alpha t + c_0$$

$$k \leq c_1 e^{-\alpha t}$$

Initial condition $\quad K(0) \leq c_1 = K_0$

$$\Rightarrow \quad K(t) \leq K_0 e^{-\frac{2\mu}{\lambda \rho_0} t} \quad \checkmark$$

In absence of fluid motion on the boundary fluid motion decays exponentially.

The rate of decay depends

$$\boxed{\nu = \frac{\mu}{\rho_0}} \quad \text{kinematic viscosity}$$