Why do we need tensors?

Scalars:

describe a quantity at a point

e.g. Temperature

Vectors:

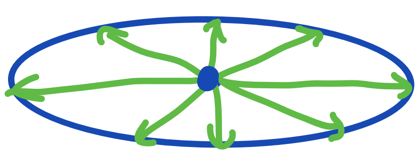
describe quantity and a direction

e.g. velocity (speed + direction) $\underline{v}$

Tensors:

describes how a quantity changes with direction

Think of an ellipsoid



Examples: anisotropic properties

stress, strain

moment of inertia

Second-order tensors

Linear operators : $\underline{v} = \underline{\underline{A}}\,\underline{u}$

maps vector $\underline{u} \in \mathcal{V}$ into vector $\underline{v} \in \mathcal{V}$

Two tensors $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are equal if

$$\underline{\underline{A}}\,\underline{v} = \underline{\underline{B}}\,\underline{v} \qquad \text{for all } \underline{v} \in \mathcal{V}$$

Zero tensor: $\underline{\underline{0}}\,\underline{v} = \underline{0}$ for all $\underline{v} \in \mathcal{V}$

Identity tensor: $\underline{\underline{I}}\,\underline{v} = \underline{v}$ for all $\underline{v} \in \mathcal{V}$

Basic algebra

α = scalar, $\underline{v}$ = vector, $\underline{\underline{A}}$ & $\underline{\underline{B}}$ 2^{nd}-ord. tensors

1) $(\alpha \underline{\underline{A}})\,\underline{v} = \underline{\underline{A}}\,(\alpha \underline{v})$ scalar multiplication

2) $(\underline{\underline{A}} + \underline{\underline{B}})\,\underline{v} = \underline{\underline{A}}\,\underline{v} + \underline{\underline{B}}\,\underline{v}$ tensor sum

3) $(\underline{\underline{A}}\,\underline{\underline{B}})\,\underline{v} = \underline{\underline{A}}\,(\underline{\underline{B}}\,\underline{v})$ tensor product

4) (tensor scalar product → later)

1 + 2 $\Rightarrow$ imply linearity

1, 2, 3 produce other tensors

set $\mathcal{V}^2$ of second order tensors $\Rightarrow$ vector space

Q: What is a basis for $\mathcal{V}^2$?

Representation of a tensor

In a frame $\{\underline{e}_i\}$ a second order tensor $\underline{\underline{S}}$ is represented by nine numbers

$$\boxed{S_{ij} = \underline{e}_i \cdot \underline{\underline{S}}\,\underline{e}_j}$$

Matrix representation of tensor in $\{\underline{e}_i\}$

$$[\underline{\underline{S}}] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \in \mathbb{R}^3 \times \mathbb{R}^3$$

Note that $[\underline{\underline{S}}]_{ij} = S_{ij}$

Consider $\underline{v} = \underline{\underline{S}}\,\underline{u}$ where $\underline{v} = v_k \underline{e}_k$, $\underline{u} = u_j \underline{e}_j$

$$v_k \underline{e}_k = \underline{\underline{S}}(u_j \underline{e}_j) = \underline{\underline{S}}\,\underline{e}_j\, u_j$$

multiply by $\underline{e}_i$ from left

$$v_k\, \underline{e}_i \cdot \underline{e}_k = \underline{e}_i \cdot \underline{\underline{S}}\,\underline{e}_j\, u_j$$

$$v_k\, \delta_{ik} = \underline{e}_i \cdot \underline{\underline{S}}\,\underline{e}_j\, u_j$$

$$v_i = (\underline{e}_i \cdot \underline{\underline{S}}\,\underline{e}_j)\, u_j \qquad \boxed{v_i = S_{ij}\, u_j}$$

Dyadic Product

The dyadic product of two vectors $\underline{a}$ and $\underline{b}$ is the 2^{nd}-order tensor $\underline{a} \otimes \underline{b}$ defined by

$$\boxed{(\underline{a} \otimes \underline{b})\underline{v} = (\underline{b} \cdot \underline{v})\,\underline{a}} \qquad \text{for all } \underline{v} \in \mathcal{V}$$

This has the form: $\underline{\underline{A}}\,\underline{v} = \alpha\, \underline{a}$

in components: $A_{ij} v_j = \alpha\, a_i$

$$\alpha = \underline{b} \cdot \underline{v} = b_j v_j$$

$$A_{ij} = [\underline{a} \otimes \underline{b}]_{ij}$$

$$\Rightarrow \quad [\underline{a} \otimes \underline{b}]_{ij}\, v_j = b_j v_j\, a_i$$

$$[\underline{a} \otimes \underline{b}]_{ij} v_j = (a_i b_j) v_j$$

$$\Rightarrow \quad [\underline{a} \otimes \underline{b}]_{ij} = a_i b_j$$

So that

$$\boxed{[\underline{a} \otimes \underline{b}] = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} = \underline{a}\, \underline{b}^T}$$

Linearity of dyadic product:

for scalars $\alpha, \beta \in \mathbb{R}$ and vectors $\underline{a}, \underline{b}, \underline{v}, \underline{w} \in \mathcal{V}$

$$\boxed{(\underline{a} \otimes \underline{b})(\alpha \underline{v} + \beta \underline{w}) = \alpha (\underline{a} \otimes \underline{b}) \underline{v} + \beta (\underline{a} \otimes \underline{b}) \underline{w}}$$

The product of two dyadic products

$$\boxed{(\underline{a} \otimes \underline{b})(\underline{c} \otimes \underline{d}) = (\underline{b} \cdot \underline{c})\, \underline{a} \otimes \underline{d}} \Rightarrow \text{HW2}$$

needed for tensor product.

Basis for $\mathcal{V}^2$

Given any frame $\{\underline{e}_i\}$ the nine dyadic products $\{\underline{e}_i \otimes \underline{e}_j\}$ form a basis for $\mathcal{V}^2$.

Any second-order tensor $\underline{\underline{S}}$ can be written as linear combination

$$\boxed{\underline{\underline{S}} = S_{ij}\, \underline{e}_i \otimes \underline{e}_j}$$

where $\quad S_{ij} = \underline{e}_i \cdot \underline{\underline{S}}\, \underline{e}_j$

Consider $\quad \underline{v} = \underline{\underline{S}}\, \underline{u} \quad$ with $\underline{v} = v_i\, \underline{e}_i$, $\underline{u} = u_k \underline{e}_k$

$$
\begin{aligned}
v_i \underline{e}_i &= S_{ij} (\underline{e}_i \otimes \underline{e}_j)(u_k \underline{e}_k) \\
&= S_{ij}\, u_k\, (\underline{e}_i \otimes \underline{e}_j) \cdot \underline{e}_k \qquad \text{apply def. of dyadic} \\
&= S_{ij}\, u_k (\underline{e}_j \cdot \underline{e}_k)\, \underline{e}_i = S_{ij}\, u_k\, \delta_{kj}\, \underline{e}_i
\end{aligned}
$$

$$v_i\, \underline{e}_i = S_{ij}\, u_j\, \underline{e}_i$$

Index notation for tensor-vector multiplication

$$\boxed{v_i = S_{ij}\, u_j} \qquad \text{used often}$$

Note: Transfer property of Kroneckerdelta

$$v_i = \delta_{ij} u_j = u_i$$

also applies to indices of tensors

for example above

$$S_{ij} u_k \delta_{kj} \underline{e}_i =$$

$$u_k \underbrace{S_{ij} \delta_{kj}}_{S_{ik}} \underline{e}_i = S_{ik} u_k \underline{e}_i$$

Tensor algebra in components

Addition : $\underline{\underline{H}} = \underline{\underline{S}} + \underline{\underline{T}}$

$$H_{ij}\, \underline{e}_i \otimes \underline{e}_j = S_{ij}\, \underline{e}_i \otimes \underline{e}_j + T_{ij}\, \underline{e}_i \otimes \underline{e}_j$$

$$= (S_{ij} + T_{ij})\, \underline{e}_i \otimes \underline{e}_j$$

$$\boxed{H_{ij} = S_{ij} + T_{ij}}$$

Scalar multipication : $\underline{\underline{H}} = \alpha\, \underline{\underline{S}} \Rightarrow \boxed{H_{ij} = \alpha\, S_{ij}}$

Product : $\underline{\underline{H}} = \underline{\underline{S}}\; \underline{\underline{T}}$

$$\underline{\underline{H}} = S_{ij} (\underline{e}_i \otimes \underline{e}_j)\, T_{kl} (\underline{e}_k \otimes \underline{e}_l)$$

$$= S_{ij} T_{kl}\, \underbrace{(\underline{e}_i \otimes \underline{e}_j)(\underline{e}_k \otimes \underline{e}_l)}_{\text{product of two dyads}}$$

$$= S_{ij} T_{kl}\, (\underline{e}_j \cdot \underline{e}_k)\, \underline{e}_i \otimes \underline{e}_l$$

$$\delta_{jk}$$

$$= S_{ij} T_{jl}\, \underline{e}_i \otimes \underline{e}_l$$

$$H_{il}\, \underline{e}_i \otimes \underline{e}_l = S_{ij} T_{jl}\, \underline{e}_i \otimes \underline{e}_l$$

$$\Rightarrow \quad \boxed{H_{il} = S_{ij} T_{jl}}$$

note the dummy j !

Determinant and Inverse

The determinat of $\underline{\underline{A}} \in \mathcal{V}^2$ is the salar

$$\boxed{\det(\underline{\underline{A}}) = \det \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = \epsilon_{ijk} [\underline{\underline{A}}]_{i1} [\underline{\underline{A}}]_{j2} [\underline{\underline{A}}]_{k3}}$$

where $[\underline{\underline{A}}]_{i1}$, $[\underline{\underline{A}}]_{j2}$, $[\underline{\underline{A}}]_{k3}$ are the columns of $[\underline{\underline{A}}]$

Triple scalar product: $(\underline{a} \times \underline{b}) \cdot \underline{c} = \det[\underline{a}, \underline{b}, \underline{c}] = \epsilon_{ijk} a_i b_j c_K$

determinants $\Rightarrow$ volumes

Properties: $\det(\underline{\underline{A}}\,\underline{\underline{B}}) = \det(\underline{\underline{A}}) \det(\underline{\underline{B}})$

$\det(\underline{\underline{A}}^T) = \det(\underline{\underline{A}})$

$\det(\alpha \underline{\underline{A}}) = \alpha^n \det(\underline{\underline{A}})$ $\quad$ ($\underline{\underline{A}}$ is $n \times n$)

$\underline{\underline{A}}$ is singular if $\det \underline{\underline{A}} = 0$.

If $\det \underline{\underline{A}} \neq 0$ then the inverse $\underline{\underline{A}}^{-1}$ exists

$$\boxed{\underline{\underline{A}}^{-1} \underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \underline{\underline{I}}}$$

Transpose of a tensor

To any $\underline{\underline{S}} \in \mathcal{V}^2$ we associate a transpose $\underline{\underline{S}}^T \in \mathcal{V}^2$ the unique tensor such that

$$\boxed{\underline{\underline{S}}\,\underline{u} \cdot \underline{v} = \underline{u} \cdot \underline{\underline{S}}^T \underline{v}} \quad \text{for all } \underline{u}, \underline{v} \in \mathcal{V}$$

This implies that $S^T_{ij} = S_{ji}$ as follows

$$(S_{ij} u_j \underline{e}_i) \cdot (v_\ell \underline{e}_\ell) = (u_k \underline{e}_k) \cdot (S^T_{ij} v_j \underline{e}_i)$$

$$S_{ij} u_j v_\ell (\underline{e}_i \cdot \underline{e}_\ell) = S^T_{ij} v_j u_k (\underline{e}_k \cdot \underline{e}_i)$$

$$S_{ij} u_j v_\ell \, \delta_{i\ell} = S^T_{ij} v_j u_k \, \delta_{ki}$$

$$S_{ij} u_j v_i = S^T_{ij} v_j u_i$$

rename indices $i \leftrightarrow j$ on rhs

$$S_{ij} u_j v_i = S^T_{ji} u_j v_i$$

$$\Rightarrow \quad S_{ij} = S^T_{ji} \quad \checkmark$$

Properties of transpose:

$$(\underline{\underline{A}}^T)^T = \underline{\underline{A}}$$

$$(\underline{\underline{A}}\,\underline{\underline{B}})^T = \underline{\underline{B}}^T \underline{\underline{A}}^T$$

$$(\underline{u} \otimes \underline{v})^T = \underline{v} \otimes \underline{u}$$

$\underline{\underline{S}}$ is symmetric if $\underline{\underline{S}} = \underline{\underline{S}}^T$ $\quad S_{ij} = S_{ji}$

$\underline{\underline{S}}$ is skew-symmetric if $\underline{\underline{S}} = -\underline{\underline{S}}^T$ $\quad S_{ij} = -S_{ji}$

Symmetric-Skew decomposition:

Any tensor $\underline{\underline{S}} \in \mathcal{V}^2$ can be written as

$$\boxed{\begin{aligned} \underline{\underline{S}} &= \underline{\underline{E}} + \underline{\underline{W}} \\ \underline{\underline{E}} &= \tfrac{1}{2}(\underline{\underline{S}} + \underline{\underline{S}}^T) \\ \underline{\underline{W}} &= \tfrac{1}{2}(\underline{\underline{S}} - \underline{\underline{S}}^T) \end{aligned}} \qquad \begin{aligned} \underline{\underline{E}} &= \underline{\underline{E}}^T \\ \underline{\underline{W}} &= -\underline{\underline{W}}^T \end{aligned}$$

Note: Skew tensors often related to rotation

Trace of a tensor

We define the trace of a dyad as

$$\mathrm{tr}(\underline{a} \otimes \underline{b}) = \underline{a} \cdot \underline{b} = a_i b_i$$

this implies that

$$\boxed{\mathrm{tr}(\underline{\underline{A}}) = A_{ii} = A_{11} + A_{22} + A_{33}}$$

as follows $\quad tr(A_{ij}\, \underline{e}_i \otimes \underline{e}_j) = A_{ij}\, tr(\underline{e}_i \otimes \underline{e}_j)$

$$= A_{ij}\, \delta_{ij} = A_{ii}$$

Properties: $tr(\underline{\underline{A}}^T) = tr(\underline{\underline{A}})$

$$tr(\underline{\underline{A}}\,\underline{\underline{B}}) = tr(\underline{\underline{B}}\,\underline{\underline{A}})$$

$$tr(\underline{\underline{A}} + \underline{\underline{B}}) = tr(\underline{\underline{A}}) + tr(\underline{\underline{B}})$$

$$tr(\alpha \underline{\underline{A}}) = \alpha\, tr(\underline{\underline{A}})$$

Decomposition: $\boxed{\underline{\underline{A}} = \alpha \underline{\underline{I}} + \text{dev}\, \underline{\underline{A}}}$

Spherical tensor: $\alpha \underline{\underline{I}}$ where $\alpha = \frac{1}{3} tr(\underline{\underline{A}})$

Deviatoric tensor: $\text{dev}\, \underline{\underline{A}} = \underline{\underline{A}} - \alpha \underline{\underline{I}}$

$$tr(\text{dev}\, \underline{\underline{A}}) = 0$$