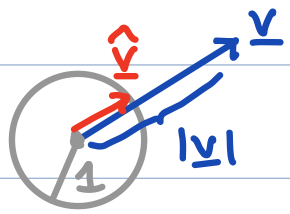
Vectors and index notation

Def: Vector $\underline{v}$ is a quantity with a magnitude and a direction

$\underline{v} = |\underline{v}|\,\hat{\underline{v}}$ $\qquad |\underline{v}| \geq 0$ magnitude (scalar)

$\hat{\underline{v}} = \dfrac{\underline{v}}{|\underline{v}|}$ direction (unit vector)



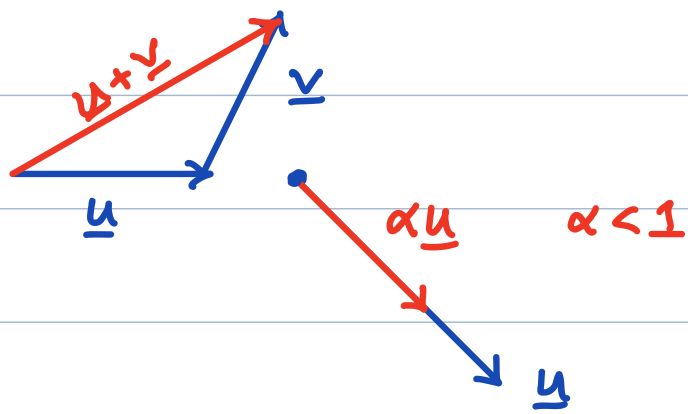
Physical examples: velocity, force, heat flux

Q: Is it possible to have a vector with out direction?

Def: Vector space, $\mathcal{V}$, is a collection of objects that is closed under addition and scalar multiplication

$\underline{u} \in \mathcal{V}$ $\qquad \underline{v} \in \mathcal{V}$ $\qquad \alpha \in \mathbb{R}$

1) $\underline{u} + \underline{v} \in \mathcal{V}$



2) $\alpha \underline{u} \in \mathcal{V}$

Q: Do vectors in $\mathbb{R}^+$ form a vector space?

Basis for a vector space

Def.: Basis for $\mathcal{V}$ is a set of linearly independent vectors $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ that span the space (3D).

Examples in 2D: $\{\underline{e}\} = \{\underline{e}_1, \underline{e}_2\}$

many choices $\Rightarrow$ not unique

We use orthonormal basis $\{\underline{e}\} = \{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$

normal: $|\underline{e}_1| = |\underline{e}_2| = |\underline{e}_3| = 1$

ortho: $\underline{e}_1 \perp \underline{e}_2$, $\underline{e}_1 \perp \underline{e}_3$, $\underline{e}_2 \perp \underline{e}_3$

(refrence) frame = orthonormal basis

Q: Any additional common restrictions on basis?

Components of a vector in a basis

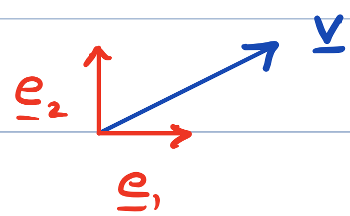
Project $\underline{v}$ onto basis vectors to get components.

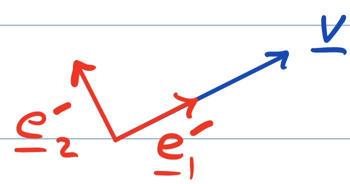
$$\underline{v} = v_1 \underline{e}_1 + v_2 \underline{e}_2 + v_3 \underline{e}_3 \qquad [\underline{v}] = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Here $[\underline{v}]$ is the representation of $\underline{v}$ in $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$

$$\boxed{\text{vector} \longleftrightarrow \text{representation}}$$

Example:





$$[\underline{v}] = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad [\underline{v}]' = \begin{bmatrix} \sqrt{5} \\ 0 \end{bmatrix}$$

$$|\underline{v}| = \sqrt{1^2 + 2^2} = \sqrt{5} \qquad |\underline{v}| = \sqrt{\sqrt{5}^2 + 0^2} = \sqrt{5}$$

The vector is the same but representation is not.

Index notation

1) Dummy index

$\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$

$$\underline{v} = v_1 \underline{e}_1 + v_2 \underline{e}_2 + v_3 \underline{e}_3 = \sum_{i=1}^{3} v_i \underline{e}_i \equiv v_i \underline{e}_i$$

If index is repeated twice in a term

$\Rightarrow$ summation is implied

(Einstein summation convention)

$\Rightarrow$ dummy index

Note: $\underline{a} = a_i \underline{e}_i = a_k \underline{e}_k = a_q \underline{e}_q$

$\Rightarrow$ rename dummy indices

2) Free index

occurs only once in a term

Example: $a_i = c_j b_j b_i$ $\quad$ j = dummy index

i = free index

$\Rightarrow$ set of equations $\quad i,j \in \{1,2,3\}$

$a_1 = \left(\sum_{j=1} c_j b_j\right) b_1$, $\quad a_2 = \left(\sum_{j=1} c_j b_j\right) b_2$, $\quad a_3 = \left(\sum_{j=1} c_j b_j\right) b_3$

Basis: $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\} = \{\underline{e}_i\}$

Note: - all terms must have same free indices
- there can be more than one free index
- same symbol cannot be used for both free and dummy index

Q: Why are these expressions meaningless?

1) $a_i = b_j$

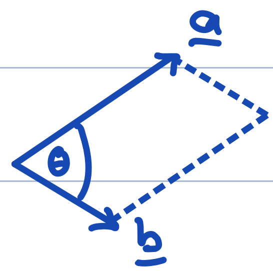
2) $a_i b_j = c_i d_j d_j$

3) $a_i b_j = c_i c_k d_k d_j + d_p c_e c_e d_q$

4) $a_i = b_k c_k d_k e_i$

Scalar product $\quad \underline{a}, \underline{b} \in \mathcal{V}$

$$\boxed{\underline{a} \cdot \underline{b} = |\underline{a}|\,|\underline{b}| \cos\theta} \qquad \theta \in [0, \pi]$$



$\underline{a} \cdot \underline{b} = 0 \qquad \underline{a} \perp \underline{b}$

$\underline{a} \cdot \underline{a} = |\underline{a}|^2$

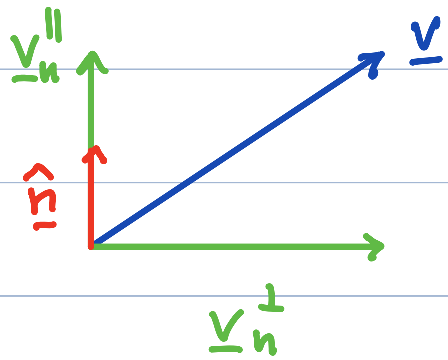
$\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$ commutative

Projection: $\hat{\underline{n}}$ = unit vector

$\underline{v} = \underline{v}_n^{\parallel} + \underline{v}_n^{\perp}$

$\underline{v}_n^{\parallel} = (\underline{v} \cdot \hat{\underline{n}})\, \hat{\underline{n}}$

$\underline{v}_n^{\perp} = \underline{v} - \underline{v}_n^{\parallel}$



$\Rightarrow$ components in a basis $\{\underline{e}_i\}$

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad \begin{aligned} v_1 &= \underline{v} \cdot \underline{e}_1 \\ v_2 &= \underline{v} \cdot \underline{e}_2 \\ v_3 &= \underline{v} \cdot \underline{e}_3 \end{aligned}$$

Kronecker Delta

To any frame $\{\underline{e}_i\}$ we associate

$$\boxed{\delta_{ij} = \underline{e}_i \cdot \underline{e}_j = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}}$$

$\Rightarrow$ orthonormal basis

δ_{ij} expresses scalar product in index notation

Properties: $\delta_{ij} = \delta_{ji}$ symmetry

$\underline{e}_i = \delta_{ij}\, \underline{e}_j$ transfer property

Examples:

Projection on basis

$\underline{u} = u_i \underline{e}_i$

$$\underline{u} \cdot \underline{e}_j = (u_i \underline{e}_i) \cdot \underline{e}_j = u_i\, \underline{e}_i \cdot \underline{e}_j = u_i\, \delta_{ij} = u_j$$

Scalar product: $\underline{a} = a_i \underline{e}_i$ $\underline{b} = b_j \underline{e}_j$

$$\underline{a} \cdot \underline{b} = (a_i \underline{e}_i) \cdot (b_j \underline{e}_j) = a_i b_j\, \underline{e}_i \cdot \underline{e}_j$$

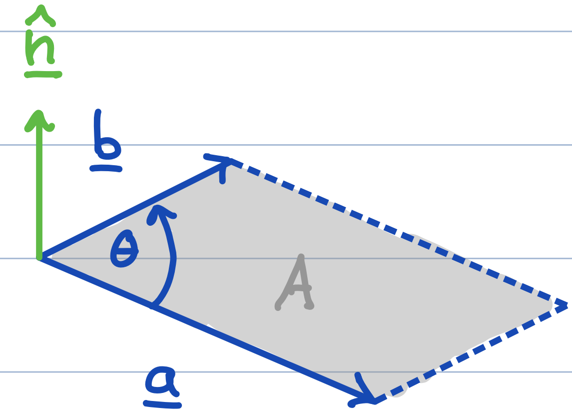
$$= a_i b_j\, \delta_{ij}$$

$$= a_i b_i = a_j b_j$$

Vector product

$\underline{a}, \underline{b} \in \mathcal{V}$

$$\boxed{\underline{a} \times \underline{b} = |\underline{a}|\,|\underline{b}| \sin\theta \; \hat{\underline{n}}}$$



$\theta \in [0, \pi]$

$\hat{\underline{n}}$ unit vector $\perp$ to $\underline{a}$ & $\underline{b}$

(right hand rule)

$|\underline{a} \times \underline{b}|$ = area of parallelogram

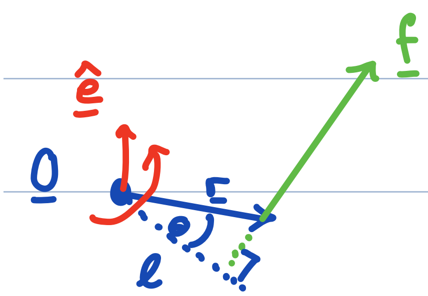
Q: Significance of $\underline{a} \times \underline{b} = \underline{0}$?

Note: $\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$ $\Rightarrow$ not commutative

Physical interpretations:

1) Moment/torque

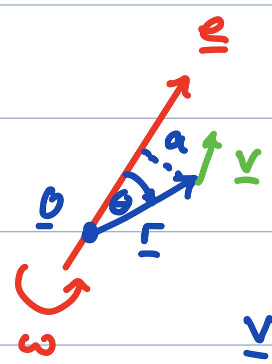
$|\tau| = \ell\,|\underline{f}|$ $\quad \ell = |\underline{r}| \sin\theta$



$\underline{\tau} = |\underline{f}|\,|\underline{r}| \sin\theta = \underline{r} \times \underline{f}$

2) Velocity of rotation

$\underline{\omega} = \omega\, \underline{e}$ $\quad \omega = |\underline{\omega}|$



$|\underline{v}| = \omega a$

$a = |r| \sin\theta$

$\underline{v} = |\underline{\omega}|\,|\underline{r}| \sin\theta$

$\underline{v} = \underline{\omega} \times \underline{r}$

Permutation symbol (Levi-Civita)

vector product in index notation

$$\boxed{\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \in \{123, 231, 312\} \text{ even perm.} \\ -1 & \text{if } ijk \in \{321, 213, 132\} \text{ odd perm.} \\ 0 & \text{repeated index} \end{cases}}$$

Flipping any 2 indices changes sign

$$\epsilon_{ijk} = -\epsilon_{kji} = -\epsilon_{jik} = -\epsilon_{ikj}$$

Invariant under cyclic permutation

$$\epsilon_{ijk} = \epsilon_{kij} = \epsilon_{jki}$$

Relation to frame $\{\underline{e}_i\}$

$$\boxed{\underline{e}_i \times \underline{e}_j = \epsilon_{ijk}\,\underline{e}_k}$$

Vector product: $\underline{a} \times \underline{b} = \underline{c}$

$\underline{a} = a_i\,\underline{e}_i \quad \underline{b} = b_j\,\underline{e}_j \quad \underline{c} = c_k\,\underline{e}_k$

$$\underline{a} \times \underline{b} = (a_i\,\underline{e}_i) \times (b_j\,\underline{e}_j) = a_i b_j\,(\underline{e}_i \times \underline{e}_j)$$

$$= \underbrace{a_i b_j\,\epsilon_{ijk}}_{c_k}\,\underline{e}_k = \underline{c} =$$

$$= c_k\,\underline{e}_k$$

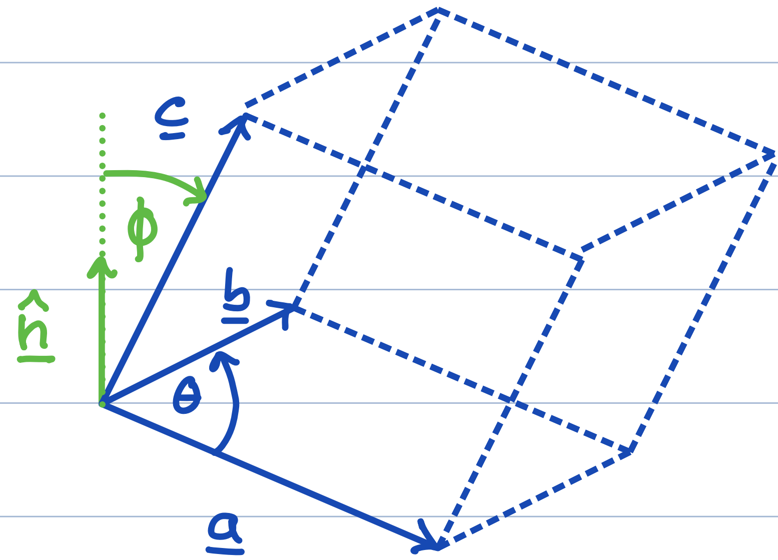
$$\Rightarrow \boxed{c_k = \epsilon_{ijk}\,a_i b_j}$$

Triple scalar product

$\underline{a}, \underline{b}, \underline{c} \in \mathcal{V}$

$$\boxed{(\underline{a} \times \underline{b}) \cdot \underline{c} = |\underline{a}|\,|\underline{b}|\,|\underline{c}| \sin\theta \cos\phi}$$

$\Rightarrow$ Volume of the parallelepiped



$(\underline{a} \times \underline{b}) \cdot \underline{c} = \underline{0} \Rightarrow$ linearly dependent

$(\underline{a} \times \underline{b}) \cdot \underline{c} > 0 \Rightarrow$ right handed

$(\underline{a} \times \underline{b}) \cdot c < 0 \Rightarrow$ left handed

Cartesian reference frame

right handed orthonormal basis $\{\underline{e}_i\}$

$\Rightarrow \quad (\underline{e}_1 \times \underline{e}_2) \cdot \underline{e}_3 = 1$

Relation to Levi-Civita

$$\boxed{\epsilon_{ijk} = (\underline{e}_i \times \underline{e}_j) \cdot \underline{e}_k}$$

Proof: $\epsilon_{ijk} = (\underline{e}_i \times \underline{e}_j) \cdot \underline{e}_k$

$= \epsilon_{ijl}\, \underline{e}_l \cdot \underline{e}_k$

$= \epsilon_{ijl}\, \delta_{lk} = \epsilon_{ijk}$ ✓

Use: $\underline{a} = a_i \underline{e}_i$, $\underline{b} = b_j \underline{e}_j$, $\underline{c} = c_k \underline{e}_k$

$$(\underline{a} \times \underline{b}) \cdot \underline{c} = \big((a_i \underline{e}_i) \times (b_j \underline{e}_j)\big) \cdot (c_k \underline{e}_k)$$

$$= a_i b_j c_k \underbrace{(\underline{e}_i \times \underline{e}_j) \cdot \underline{e}_k}_{\epsilon_{ijk}}$$

$$= \epsilon_{ijk} a_i b_j c_k$$

$$\boxed{(\underline{a} \times \underline{b}) \cdot \underline{c} = \epsilon_{ijk} a_i b_j c_k}$$

$\Rightarrow$ Invariant under cyclic perm.

$$(\underline{a} \times \underline{b}) \cdot \underline{c} = (\underline{c} \times \underline{a}) \cdot \underline{b} = (\underline{b} \times \underline{c}) \cdot \underline{a}$$

Relationship to determinant

matrix $[\underline{a}\ \underline{b}\ \underline{c}] = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$$\boxed{(\underline{a} \times \underline{b}) \cdot \underline{c} = \det([\underline{a}\ \underline{b}\ \underline{c}])}$$

determinants $\Rightarrow$ volumes

$$\underline{c} \cdot (\underline{a} \times \underline{b}) = \underline{a} \cdot (\underline{b} \times \underline{c})$$

$$\underline{b} \times \underline{c} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \underline{e}_1(b_2c_3 - b_3c_2) - \underline{e}_2(b_1c_3 - b_3c_1) + \underline{e}_3(b_1c_2 - b_2c_1)$$

taking dot product with $\underline{a}$ replaces first row

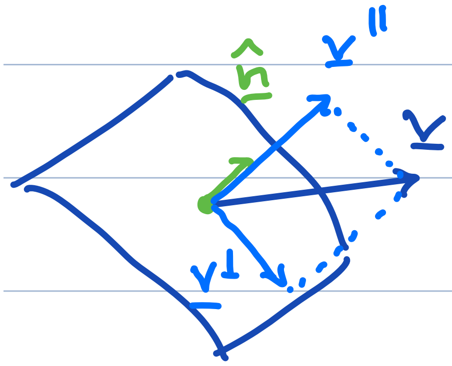
$$\Rightarrow \quad \underline{a} \cdot (\underline{b} \times \underline{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Triple vector product

$\underline{a}, \underline{b}, \underline{c} \in \mathcal{V}$

$$\boxed{\begin{aligned} \underline{a} \times (\underline{b} \times \underline{c}) &= (\underline{a} \cdot \underline{c})\, \underline{b} - (\underline{a} \cdot \underline{b})\, \underline{c} \\ (\underline{a} \times \underline{b}) \times \underline{c} &= (\underline{c} \cdot \underline{a})\, \underline{b} - (\underline{c} \cdot \underline{b})\, \underline{a} \end{aligned}}$$

$\Rightarrow$ find normal component



$$\underline{v} = \underline{v}^{\parallel} + \underline{v}^{\perp}$$

$$\underline{v}^{\parallel} = (\underline{v} \cdot \hat{\underline{n}})\, \hat{\underline{n}}$$

$$\underline{v}^{\perp} = \underline{v} - \underline{v}^{\parallel} =$$

$$\underline{v}^{\perp} = -(\underline{v} \times \hat{\underline{n}}) \times \hat{\underline{n}}$$

Epsilon-delta Identities

In any Cartesian reference frame

$$\boxed{\begin{aligned} \epsilon_{pqs}\, \epsilon_{nrs} &= \delta_{pn}\, \delta_{qr} - \delta_{pr}\, \delta_{qn} \\ \epsilon_{pqs}\, \epsilon_{rqs} &= 2\, \delta_{pr} \end{aligned}}$$

$\Rightarrow$ vector identities with two cross products

Example: $\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a}\cdot\underline{c})\underline{b} - (\underline{a}\cdot\underline{b})\underline{c} \equiv \underline{d}$

$\underline{a} = a_q \underline{e}_q, \quad \underline{b} = b_i \underline{e}_i, \quad \underline{c} = c_j \underline{e}_j, \quad \underline{d} = d_p \underline{e}_p$

$\underline{b} \times \underline{c} = \epsilon_{ijk} b_i c_j \underline{e}_k$

$$(a_q \underline{e}_q) \times (\epsilon_{ijk} b_i c_j \underline{e}_k) = \epsilon_{ijk} a_q b_i c_j \underbrace{(\underline{e}_q \times \underline{e}_k)}_{\epsilon_{qkp}\underline{e}_p}$$

$$= \epsilon_{ijk}\epsilon_{qkp} a_q b_i c_j \underline{e}_p$$

$$= \epsilon_{ijk}\epsilon_{pqk} a_q b_i c_j \underline{e}_p$$

$$= (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}) a_q b_i c_j \underline{e}_p$$

$$= a_j b_i c_j \underline{e}_i - a_i b_i c_j \underline{e}_j$$

$$= a_j c_j \, b_i \underline{e}_i - a_i b_i \, c_j \underline{e}_j$$

$$= (\underline{a}\cdot\underline{c})\,\underline{b} - (\underline{a}\cdot\underline{b})\,\underline{c}$$

Frame Identities

frame $\{\underline{e}_i\}$ = orthonormal basis

$\boxed{\underline{e}_i = \delta_{ij}\underline{e}_j}$ and $\boxed{\underline{e}_i \times \underline{e}_j = \epsilon_{ijk}\underline{e}_k}$