Polar Decomposition

Any $\underline{\underline{F}}$ with $\det \underline{\underline{F}} = \frac{dV_x}{dV_X} > 0$ has right & left polar decomposition $\boxed{\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{V}}\,\underline{\underline{R}}}$

where

$\underline{\underline{R}}$ = rotation

$$\left.\boxed{\begin{array}{l} \underline{\underline{U}} = \sqrt{\underline{\underline{F}}^T\underline{\underline{F}}} \\ \underline{\underline{V}} = \sqrt{\underline{\underline{F}}\,\underline{\underline{F}}^T} \end{array}}\right\}$$ symmetric positive definite

$\underline{a} \cdot \underline{\underline{U}}\,\underline{a} > 0$

for all $\underline{a} \in \mathcal{V}$

$\underline{a} \cdot \underline{\underline{V}}\,\underline{a} > 0$

$\Rightarrow$ eigenvalues of $\underline{\underline{U}}$ and $\underline{\underline{V}}$ are positive: $\lambda_i > 0$

note to self: yes there will be λ's!

$$\underline{\underline{F}}\,\underline{\underline{U}}^{-1} = \underline{\underline{R}}\,\underline{\underline{U}}\,\underline{\underline{U}}^{-1} \quad \Rightarrow \quad \boxed{\underline{\underline{R}} = \underline{\underline{F}}\,\underline{\underline{U}}^{-1}}$$

For symmetric tensors: $\underline{\underline{A}} = \underline{\underline{A}}^T \quad \Rightarrow \quad \boxed{(\underline{\underline{A}}^T)^{-1} = (\underline{\underline{A}}^{-1})^T}$

$$\underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{I}} \qquad\qquad \underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \underline{\underline{I}}$$

$$1 \quad (\underline{\underline{A}}^T)^{-1}\underline{\underline{A}} = \underline{\underline{I}} \qquad\qquad (\underline{\underline{A}}\,\underline{\underline{A}}^{-1})^T = \underline{\underline{I}}^T$$

$$(\underline{\underline{A}}^{-1})^T\underline{\underline{A}}^T = \underline{\underline{I}}$$

$$2 \quad (\underline{\underline{A}}^{-1})^T\underline{\underline{A}} = \underline{\underline{I}}$$

$$1+2: \quad (\underline{\underline{A}}^T)^{-1}\underline{\underline{A}} = (\underline{\underline{A}}^{-1})^T\underline{\underline{A}} \quad \Rightarrow \quad (\underline{\underline{A}}^T)^{-1} = (\underline{\underline{A}}^{-1})^T$$

Show $\underline{\underline{R}}$ is rotation:

1) $\underline{\underline{R}}^T \underline{\underline{R}} = \underline{\underline{R}}\,\underline{\underline{R}}^T = \underline{\underline{I}}$ (orthonormal)

2) $\det(\underline{\underline{R}}) = 1$ (rotation)

1) Orthonormal

$$\underline{\underline{R}}^T \underline{\underline{R}} = (\underline{\underline{F}}\,\underline{\underline{U}}^{-1})^T (\underline{\underline{F}}\,\underline{\underline{U}}^{-1}) = (\underline{\underline{U}}^{-1})^T \underline{\underline{F}}^T \underline{\underline{F}}\,\underline{\underline{U}}^{-1} = \underset{\substack{\uparrow \\ \text{sym.}}}{(\underline{\underline{U}}^T)^{-1}} \underbrace{\underline{\underline{F}}^T \underline{\underline{F}}}_{\underline{\underline{U}}^2} \underline{\underline{U}}^{-1}$$

$$= \underline{\underline{U}}^{-1} \underline{\underline{U}}^2 \underline{\underline{U}}^{-1} = \underbrace{\underline{\underline{U}}^{-1} \underline{\underline{U}}}_{\underline{\underline{I}}}\ \underbrace{\underline{\underline{U}}\,\underline{\underline{U}}^{-1}}_{\underline{\underline{I}}} = \underline{\underline{I}} \checkmark$$

2) Rotation

$$\det(R) = \det(\underline{\underline{F}}\,\underline{\underline{U}}^{-1}) = \frac{\det(\underline{\underline{F}})}{\det(\underline{\underline{U}})} > 0$$

$$\det(\underline{\underline{F}}) = \frac{dV_x}{dV_X} > 0 \quad \text{for admissible deformation}$$

$$\det(\underline{\underline{U}}) = \lambda_1 \lambda_2 \lambda_3 > 0 \quad \text{if } \underline{\underline{U}} \text{ is s.p.d.}$$

Show $\underline{\underline{U}}$ is symmetric positive definite

$|\underline{a}|^2 = \underline{a} \cdot \underline{a} > 0 \qquad \underline{a} = \underline{\underline{F}}\,\underline{v}$

$(\underline{\underline{F}}\,\underline{v}) \cdot (\underline{\underline{F}}\,\underline{v}) > 0$

$= \underline{v} \cdot \underline{\underline{F}}^T \underline{\underline{F}}\,\underline{v} = \underline{v} \cdot \underline{\underline{U}}^2 \underline{v} = \underline{v} \cdot \underline{\underline{C}}\,\underline{v} > 0$

$\Rightarrow \quad \underline{\underline{F}}^T \underline{\underline{F}} = \underline{\underline{U}}^2 = \underline{\underline{C}}$ is s.p.d.

eigenvalues of $\underline{\underline{C}}$: $\mu_i > 0$

yes that's what they will be called!

Is $\underline{\underline{U}} = \sqrt{\underline{\underline{C}}}$ also s.p.d.?

Tensor square root

If $\underline{\underline{C}}$ is a s.p.d. tensor with eigenpair $(\mu_i, \underline{u}_i)$

$$\boxed{\underline{\underline{U}} = \sqrt{\underline{\underline{C}}} = \sum_{i=1}^{3} \sqrt{\mu_i}\; \underline{u}_i \otimes \underline{u}_i}$$

eigenpair of $\underline{\underline{U}}$ is $(\lambda, \underline{u})$ where $\lambda_i = \sqrt{\mu_i} > 0$

$\Rightarrow \quad \underline{\underline{U}} = \sqrt{\underline{\underline{C}}} = \sqrt{\underline{\underline{F}}^T \underline{\underline{F}}}$ is s.p.d.

Similarly $\underline{\underline{V}} = \sqrt{\underline{\underline{F}}\,\underline{\underline{F}}^T}$ is s.p.d.

Analysis of local deformation

locally any $\varphi(\underline{X})$ can be approximated as homogeneous $\boxed{\underline{x} = \varphi(\underline{X}) = \underline{c} + \underline{\underline{F}}\,\underline{X}}$ $\qquad \underline{\underline{F}} = \nabla\varphi$

$\underline{\underline{F}}$ is measure of strain but contains rotations

$\Rightarrow$ not suitable as strain tensor

Build strain tensor in 3 steps:

1) Remove translations
2) Remove rotations
3) Find principal streches

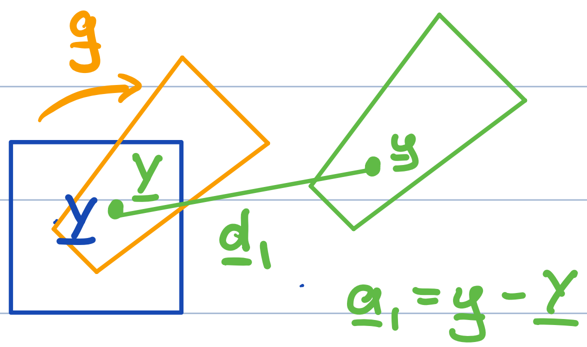
1) Translation - fixed point decomposition

Any hom. φ can be decomposed into

$$\boxed{\varphi = \underline{d}_1 \circ \underline{g} = \underline{g} \circ \underline{d}_2}$$

$\underline{g} = \underline{Y} + \underline{\underline{F}}(\underline{X} - \underline{Y})$ hom. def. with fixed point $\underline{Y}$

$\underline{d}_i = \underline{X} + \underline{a}_i$ translations from $\underline{Y}$ $i = \{1, 2\}$



$\underline{g}(\underline{x}) = \underline{Y} + \underline{\underline{F}}(\underline{x} - \underline{y})$

$\underline{d}_1(\underline{x}) = \underline{X} + \underbrace{\left(\varphi(\underline{Y}) - \underline{Y}\right)}_{\underline{a}_1}$

Rewrite φ as Taylor expansion around $\underline{Y}$

$\underline{x} = \varphi(\underline{X}) = \underline{c} + \underline{\underline{F}}\,\underline{X}$ $\qquad$ $\underline{y} = \varphi(\underline{Y}) = \underline{c} + \underline{\underline{F}}\,\underline{Y}$

$\underline{x} - \underline{y} = \underline{\underline{F}}(\underline{X} - \underline{Y})$

$\underline{x} = \underline{y} + \underline{\underline{F}}(\underline{X} - \underline{Y})$ where $\underline{x} = \varphi(\underline{X})$ and $\underline{y} = \varphi(\underline{Y})$

$$\boxed{\varphi(\underline{X}) = \varphi(\underline{Y}) + \underline{\underline{F}}(\underline{X} - \underline{Y})} \iff \varphi(\underline{X}) = \underline{c} + \underline{\underline{F}}\,\underline{X}$$

use to show decomposition $\varphi = \underline{d}_1 \circ \underline{g}$

$\underline{d}_1 = \underline{X} + \underline{\varphi}(\underline{Y}) - \underline{Y}$ and $\underline{g} = \underline{Y} + \underline{\underline{F}}(\underline{X} - \underline{Y})$

substitute:

$$\underline{\varphi}(\underline{X}) = \underline{d}_1(\underline{g}(\underline{X})) = \underline{g}(\underline{X}) + \underline{\varphi}(\underline{Y}) - \underline{Y}$$
$$= \cancel{\underline{Y}} + \underline{\underline{F}}(\underline{X} - \underline{Y}) + \underline{\varphi}(\underline{Y}) - \cancel{\underline{Y}}$$

$$\underline{\varphi}(\underline{X}) = \underline{\varphi}(\underline{Y}) + \underline{\underline{F}}(\underline{X} - \underline{Y})$$ ✓

$\Rightarrow$ proves $\underline{\varphi} = \underline{d}_1 \circ \underline{g}$

$\Rightarrow$ we can always extract translation and assume def. with fixed point!

Stretch-Rotation decomposition

For $\underline{\varphi}(\underline{X}) = \underline{Y} + \underline{\underline{F}}(\underline{X} - \underline{Y})$ with $\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{V}}\,\underline{\underline{R}}$

$$\boxed{\underline{\varphi} = \underline{r} \circ \underline{s}_1 = \underline{s}_2 \circ \underline{r}}$$

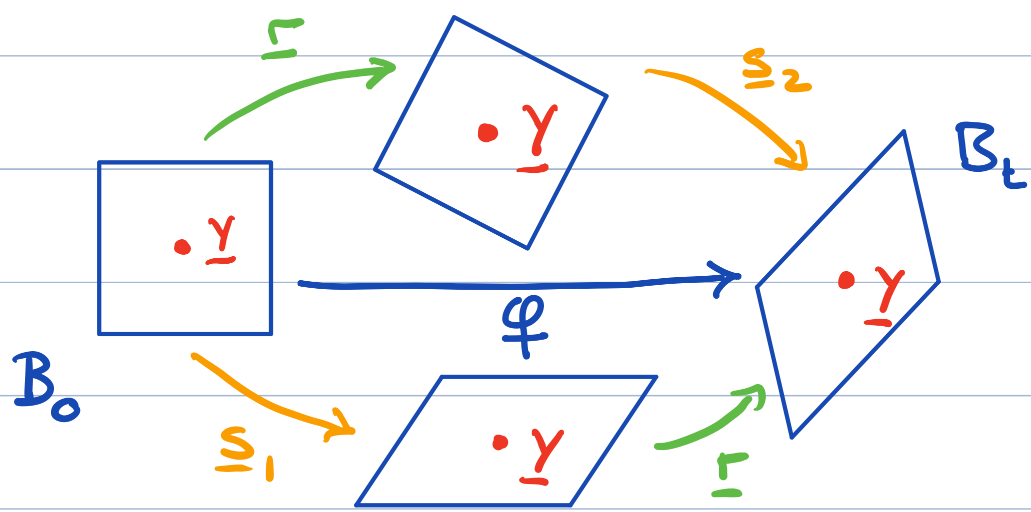
where

$\underline{r}(\underline{X}) = \underline{Y} + \underline{\underline{R}}(\underline{X} - \underline{Y})$ rotation around $\underline{Y}$

$$\left.\begin{array}{l} \underline{s}_1(\underline{X}) = \underline{Y} + \underline{\underline{U}}(\underline{X} - \underline{Y}) \\ \underline{s}_2(\underline{X}) = \underline{Y} + \underline{\underline{V}}(\underline{X} - \underline{Y}) \end{array}\right\}$$ streches from $\underline{Y}$

where $\underline{\underline{U}} = \sqrt{\underline{\underline{F}}^T \underline{\underline{F}}}$ and $\underline{\underline{V}} = \sqrt{\underline{\underline{F}}\,\underline{\underline{F}}^T}$

Graphical Interpretation



This can be shown as follows

$$(\underline{r}\circ\underline{s}_1)(\underline{X}) = \underline{r}(\underline{s}_1(\underline{X})) = \underline{Y} + \underline{\underline{R}}(\underline{s}_1(\underline{X}) - \underline{Y})$$

$$= \underline{Y} + \underline{\underline{R}}(\cancel{\underline{Y}} + \underline{U}(\underline{X}-\underline{Y}) - \cancel{\underline{Y}})$$

$$= \underline{Y} + \underline{\underline{R}}\,\underline{\underline{U}}(\underline{X}-\underline{Y}) =$$

$$= \underline{Y} + \underline{\underline{F}}(\underline{X}-\underline{Y})$$

$$(\underline{r}\circ\underline{s}_1)(\underline{X}) = \varphi(\underline{X}) \quad ✓$$

The proof for $\varphi = \underline{s}_2 \circ \underline{r}$ is similar.

Strech tensors

$\Rightarrow$ part of $\underline{F}$ that contains deformation

both $\underline{\underline{U}} = \sqrt{\underline{\underline{F}}^T \underline{\underline{F}}}$ and $\underline{\underline{V}} = \sqrt{\underline{\underline{F}}\,\underline{\underline{F}}^T}$ are s.p.d.

$\Rightarrow$ spectral decomposition

$$\boxed{\underline{\underline{U}} = \sum_{i=1}^{3} \lambda_i \, \underline{u}_i \otimes \underline{u}_i} \quad \text{and} \quad \boxed{\underline{\underline{V}} = \sum_{i=1}^{3} \lambda_i \, \underline{v}_i \otimes \underline{v}_i}$$

$(\lambda_i, \underline{u}_i)$ eigenpair of $\underline{\underline{U}}$

$(\lambda_i, \underline{v}_i)$ eigenpair of $\underline{\underline{V}}$

$\Rightarrow$ same eigen values different eigenvectors

1) Show eigenvalues are same

$$\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{V}}\,\underline{\underline{R}} \Rightarrow$$

$$\underline{\underline{R}}^T \underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{R}}^T \underline{\underline{V}}\,\underline{\underline{R}} \Rightarrow \underline{\underline{U}} = \underline{\underline{R}}^T \underline{\underline{V}}\,\underline{\underline{R}}$$ change in basis

char. polynomial:

$$p_{\underline{\underline{U}}}(\lambda) = \det(\underline{\underline{U}} - \lambda \underline{\underline{I}}) = \det(\underline{\underline{R}}^T \underline{\underline{V}}\,\underline{\underline{R}} - \lambda \underline{\underline{R}}^T \underline{\underline{R}})$$

$$= \det(\underline{\underline{R}}^T (\underline{\underline{V}} - \lambda \underline{\underline{I}})\, \underline{\underline{R}})$$

$$= \cancel{\det(\underline{\underline{R}}^T)}^{1} \det(\underline{\underline{V}} - \lambda \underline{\underline{I}}) \cancel{\det(\underline{\underline{R}})}^{1}$$

$$= \det(\underline{\underline{V}} - \lambda \underline{\underline{I}}) = p_{\underline{\underline{V}}}(\lambda) \checkmark$$

$\Rightarrow$ $\underline{\underline{U}}$ and $\underline{\underline{V}}$ same eigenvalues

λ_p's are principal streches

$\underline{u}_p$ and $\underline{v}_p$ are right and left principal directions

What is relation between $\underline{u}$ and $\underline{v}$

$\underline{\underline{U}}\,\underline{u}_p = \lambda_p \underline{u}_p$ no sum !

$\underline{\underline{R}}\,\underline{\underline{U}}\,\underline{u}_p = \lambda_p \underline{\underline{R}}\,\underline{u}_p$ $\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{V}}\,\underline{\underline{R}}$

$\underline{\underline{V}}\,\underline{\underline{R}}\,\underline{u}_p = \lambda_p \underline{\underline{R}}\,\underline{u}_p$ compare $\underline{\underline{V}}\,\underline{v}_p = \lambda_p \underline{v}_p$ no sum

$\Rightarrow$ $\boxed{\underline{v}_p = \underline{\underline{R}}\,\underline{u}_p}$ differ by rotation

Summary

Any hom. def. φ can be decomposed into sequence of three elementary deformations:

1) Translation: $\underline{d}_i$ (no strain)
2) Rotation: $\underline{r}$ (no strain)
3) Strech along principal directions: $\underline{s}_i$ ($\Rightarrow$ strain)

$$\boxed{\varphi = \underline{r} \circ \underline{s}_1 \circ \underline{d}_1 = \underline{d}_2 \circ \underline{s}_2 \circ \underline{r}}$$

Cauchy-Green Strain Tensor

Consider a deformation $\varphi : \mathcal{B} \to \mathcal{B}'$ with $\underline{\underline{F}} = \nabla \varphi$, then the (right) Cauchy-Green strain tensor is

$$\boxed{\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}}} .$$

Note that $\underline{\underline{C}}$ is always symmetric pos. definite.

The deformation gradient $\underline{\underline{F}}$ contains information about both rotations and streches. Using the right polar decomposition we have

$$\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}}$$

$\underline{\underline{R}}$ is rotation matrix

$\underline{\underline{U}} = \sqrt{\underline{\underline{F}}^T \underline{\underline{F}}}$ is right strech tensor

Clearly $\underline{\underline{C}} = \underline{\underline{U}}^2$ and the rotation $\underline{\underline{R}}$ implicit in $\underline{\underline{F}}$ is not present in in $\underline{\underline{C}}$.

$\Rightarrow$ **The right Cauchy Green strain tensor only contains information about streches.**

Hence we can cannot obtain $\underline{\underline{F}}$ from $\underline{\underline{C}}$!

Remarks:

1) Stricktly the right-stretch tensor $\underline{\underline{U}}$ is sufficient. We introduce $\underline{\underline{C}} = \underline{\underline{U}}^2$ to avoid the tensor square root.

Simple example:

$$[\underline{\underline{F}}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$[\underline{\underline{C}}] = [\underline{\underline{F}}^T][\underline{\underline{F}}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 4 \\ 0 & 4 & 5 \end{pmatrix}$$

To get $[\underline{\underline{U}}]$ we need to solve eigenvalue problem

$$\begin{vmatrix} 1-\mu & 0 & 0 \\ 0 & 5-\mu & 4 \\ 0 & 4 & 5-\mu \end{vmatrix} = (1-\mu)(5-\mu)^2 - 16(1-\mu) = 0$$

Eigenvalues: $\mu_{1,2} = 1 \qquad \mu_3 = 9$

Eigenvectors: $[\underline{u}_1] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad [\underline{u}_2] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \quad \underline{u}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Hence: $$[\underline{\underline{U}}] = \sqrt{[\underline{\underline{C}}]} = \sum_{i=1}^{3} \sqrt{\mu_i}\, \underline{u}_i \otimes \underline{u}_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

2) $\underline{\underline{U}} = \sum_{i=1}^{3} \lambda_i \, \underline{u}_i \otimes \underline{u}_i$ where

λ_i's are principal stretches

$\underline{u}_i$'s are right principal directions

$\underline{\underline{C}} = \underline{\underline{U}}^2 = \sum_{i=1}^{3} \lambda_i^2 \, \underline{u}_i \otimes \underline{u}_i$

$\mu_i = \lambda_i^2$ eig. values of $\underline{\underline{C}}$ are squares of principal streches

eigenvectors are right principal dir.

3) $C_{KL} = F_{iK} F_{iL}$ "material strain tensor"

spatial indices are contracted

Other strain tensors

I) $\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}})$: Green-Lagrange tensor

$E_{KL} = \frac{1}{2}(C_{KL} - \delta_{KL})$ material tensor $\Rightarrow$ linear theory

II) $\underline{\underline{b}} = \underline{\underline{F}}\,\underline{\underline{F}}^T = \underline{\underline{V}}^2$: Left Cauchy-Green tensor

$b_{kl} = F_{kI} F_{lI}$ "spatial tensor" (a.k.a. Finger tensor)

III) $\underline{\underline{e}} = \frac{1}{2}(\underline{\underline{I}} - \underline{\underline{F}}^{-T}\underline{\underline{F}}^{-1})$: Euler-Almansi tensor

$e_{kl} = \frac{1}{2}(\delta_{kl} - F^{-1}_{Ik} F^{-1}_{Il})$ "spatial tensor"