Cauchy Stress Tensor

Derivation:

0) Start from Newton's second law
1) Show surface forces dominat $V \to 0$
2) Cauchy postulate (local)
3) Law of action & reaction (Newton's 3^{rd} law)
4) Cauchy's theorem

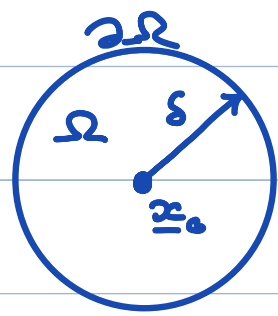
$\Rightarrow$ Cauchy stress

1) Force balance as $V \to 0$

2^{nd} law: $\underline{f} = m\,\underline{a}$

resultant forces: $\underline{f} = \underline{F}_b + \underline{F}_s$

$$\Rightarrow \boxed{\underline{F}_s = m\,\underline{a} - \underline{F}_b}$$



mass: $m = \int_\Omega \rho \, dV$

body force: $\underline{F}_b = \int_\Omega \underline{b} \, dV$

surface force: $\underline{F}_s = \oint_{\partial\Omega} \underline{t} \, dA$

V_Ω = volume

$A_{\partial\Omega}$ = area

Limit of vanishing body

$$\lim_{\delta \to 0} \underline{F}_s = m\,\underline{a} - \underline{F}_b \; ?$$

mass: $$\lim_{\delta \to 0} m = \int_\Omega \rho(\underline{x})\, dV \approx \rho(\underline{x}_0) \int_\Omega dV = \underline{\rho_0 V_\Omega}$$

body force: $$\lim_{\delta \to 0} \underline{F}_b = \int_\Omega \rho g\, dV \approx \rho(\underline{x}_0)\, g \int_\Omega dV = \underline{\rho_0 g V_\Omega}$$

substitute into 2nd law

$$\lim_{\delta \to 0} \oint_{\partial\Omega} \underline{t}\, dA = \int_\Omega \rho \underline{a} - \rho g\, dV \approx \rho_0 (\underline{a} - g)\, V_\Omega$$

normalize by surface area!

$$\lim_{\delta \to 0} \frac{1}{A_{\partial\Omega}} \oint_{\partial\Omega} \underline{t}\, dA = \frac{V_\Omega}{A_{\partial\Omega}} \rho_0 (\underline{a} - g)$$

Volume vanishes faster than surface area!

$$\lim_{\delta \to 0} \frac{V_\Omega}{A_{\partial\Omega}} = 0$$

Consider a sphere: $V_\Omega = \frac{4}{3}\pi\delta^3 \quad A_{\partial\Omega} = 4\pi\delta^2$

$$\lim_{\delta \to 0} \frac{V_\Omega}{A_{\partial\Omega}} = \frac{\delta}{3} = 0$$ (holds for any shape)

Surface forces vanish of infinitesimal body

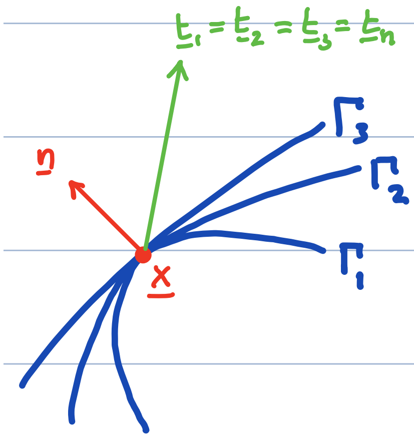
$$\boxed{\lim_{\delta \to 0} \frac{1}{A_\Omega} \oint_{\partial\Omega} \underline{t} \, dA = 0}$$

Note: • $\frac{1}{A_\Omega}$ normalization

• assume ρ, $|\underline{g}|$ and $|\underline{a}|$ are finite

$\Rightarrow$ basis for derivation of Cauchy stress tensor.

2) Cauchy's Postulate



Traction depends point wise on normal $\Rightarrow$ local

$$\boxed{\underline{t} = \underline{t}(\underline{n}(\underline{x}), \underline{x})}$$

Note: there are non-local theories!

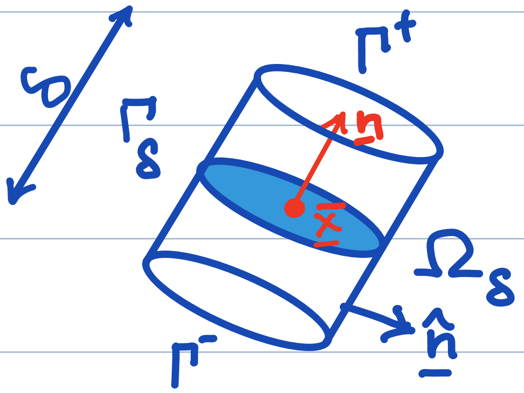
$\underline{t} = \underline{t}(\underline{n}, \nabla \underline{n}, \underline{x})$ advanced stuff

curvature of Γ

3. Law of Action and Reaction

$\boxed{\underline{t}(-\underline{n},\underline{x}) = -\underline{t}(\underline{n},\underline{x})}$ $\quad$ $\underline{t}$ is continuous & bounded

Proof: Force balance on cylinder



$\partial\Omega_\delta = \Gamma^+ \cup \Gamma^- \cup \Gamma_\delta$

on Γ^+: $\hat{\underline{n}} = \underline{n}$

on Γ^-: $\hat{\underline{n}} = -\underline{n}$

$\Gamma^\pm = D$ area of disk

$\lim\limits_{\delta\to 0} \Gamma_\delta = 0$

Surface force:

$$\underline{F}_\delta = \int_{\partial\Omega_\delta} \underline{t}(\underline{n},\underline{x})\, dA$$

$$\underline{F}_\delta = \int_{\Gamma_\delta} \underline{t}(\hat{\underline{n}},\underline{x})\, dA + \int_{\Gamma^+} \underline{t}(\underline{n},\underline{x})\, dA + \int_{\Gamma^-} \underline{t}(-\underline{n},\underline{x})\, dA$$

$$\lim_{\delta\to 0} \frac{1}{\underset{\substack{\uparrow \\ 2D}}{A_{\partial\Omega_\delta}}} \overset{0}{\cancel{\int_{\Gamma_\delta} \underline{t}(\hat{\underline{n}},x)\, dA}} + \underbrace{\int_{\Gamma^+} \underline{t}(\underline{n},x)\, dA}_{D\, \underline{t}(\underline{n},\bar{\underline{x}})} + \underbrace{\int_{\Gamma^-} \underline{t}(-\underline{n},x)\, dA}_{D\, \underline{t}(-\underline{n},\bar{\underline{x}})} = 0$$

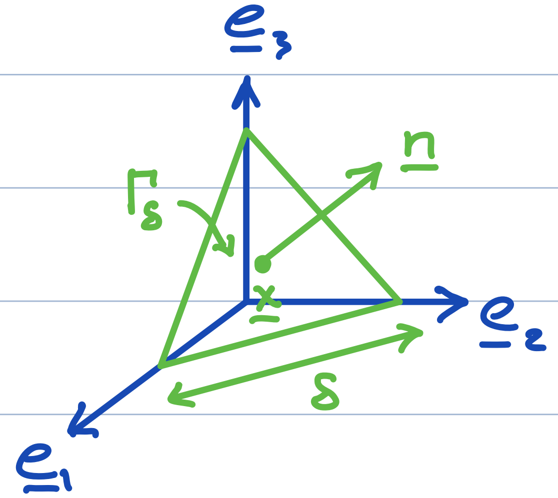
$$\Rightarrow \quad \underline{t}(\underline{n},\bar{\underline{x}}) + \underline{t}(-\underline{n},\underline{x}) = \underline{0} \quad \checkmark$$

Cauchy's Theorem

If $\underline{t}(\underline{n},\underline{x})$ satisfies Cauchy's postulate then $\underline{t}(\underline{n},\underline{x})$ is linear in $\underline{n}$

$\Rightarrow$ $\boxed{\underline{t}(\underline{n},\underline{x}) = \underline{\underline{\sigma}}(\underline{x})\,\underline{n}}$ $\qquad \underline{\underline{\sigma}}(\underline{x})$ Cauchy stress

Proof:



Ω_δ tetrahedron

$\partial\Omega_\delta = \Gamma_\delta \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$

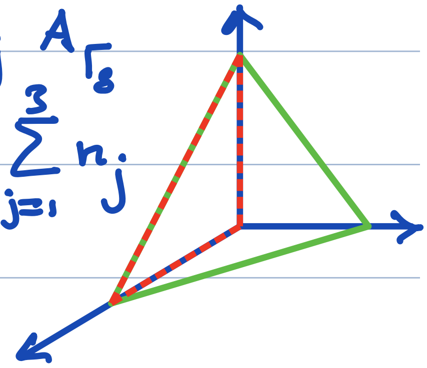
Γ_δ: $\quad \underline{n}\cdot\underline{e}_i > 0.$

Γ_i: $\quad \underline{n}_j = -\underline{e}_j$

$$\lim_{\delta\to 0} \frac{1}{A_{\partial\Omega_\delta}} \int_{\partial\Omega_\delta} \underline{t}(\underline{n}(\underline{y}),\underline{y})\, dA = 0$$

$$\lim_{\delta\to 0} \frac{1}{A_{\partial\Omega_\delta}} \left[\int_{\Gamma_\delta} \underline{t}(\underline{n},\underline{y})\, dA + \sum_{j=1}^{3} \int_{\Gamma_j} \underline{t}(-\underline{e}_j,\underline{y})\, dA \right] = 0$$

Homework 1 you showed: $A_{\Gamma_j} = n_j A_{\Gamma_\delta}$



$$A_{\partial\Omega_\delta} = A_{\Gamma_\delta} + \sum_{j=1}^{3} A_{\Gamma_j} = \lambda A_{\Gamma_\delta} \qquad \lambda = 1 + \sum_{j=1}^{3} n_j$$

substituting

$$\lim_{\delta\to 0} \frac{1}{A_{\partial\Omega_\delta}} \left[\int_{\Gamma_\delta} \underline{t}(\underline{n}, \underline{y})\, dA + \sum_{j=1}^{3} \int_{\Gamma_\delta} t_n(-\underline{e}_j, \underline{y})\, n_j\, dA \right] = \underline{0}$$

$$\lim_{\delta\to 0} \frac{1}{\lambda A_{\Gamma_\delta}} \int_{\Gamma_\delta} \underbrace{\underline{t}(\underline{n}, \underline{y}) + \sum_{j=1}^{3} t(-\underline{e}_j, \underline{y})\, n_j}_{f(y)}\, dA = \underline{0}$$

$$\lim_{\delta\to 0} \frac{1}{\lambda A_{\Gamma_\delta}} \int_{\Gamma_\delta} f(\underline{y})\, dA = \frac{1}{\lambda A_{\Gamma_\delta}} f(\underline{x})\, A_{\Gamma_\delta} = \frac{1}{\lambda} f(\underline{x}) = 0$$

$$\Rightarrow f(x) = 0$$

$$\Rightarrow \underline{t}(\underline{n}, \underline{x}) + \sum_{j=1}^{3} t(-\underline{e}_j, \underline{x})\, n_j = \underline{0}$$

$$\underline{t}(\underline{n}, \underline{x}) = -\sum_{j=1}^{3} \underline{t}(-\underline{e}_j, \underline{x})\, n_j$$

using action & reaction:

$$\underline{t}(\underline{n}, \underline{x}) = \sum_{j=1}^{3} \underline{t}(\underline{e}_j, \underline{x})\, n_j$$

traction is weighted sum of tractions on coord. planes

$$\underline{t}(\underline{n},\underline{x}) = \underline{t}(\underline{e}_1,\underline{x})n_1 + \underline{t}(\underline{e}_2,\underline{x})n_2 + \underline{t}(\underline{e}_3,\underline{x})n_3$$

$$= \underbrace{\left[\underline{t}(\underline{e}_1,\underline{x}), \underline{t}(\underline{e}_2,\underline{x}), \underline{t}(\underline{e}_3,\underline{x})\right]}_{\underline{\underline{\sigma}}(\underline{x})} \underbrace{\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}}_{\underline{n}}$$

$$\Rightarrow \quad \underline{t}(\underline{n},\underline{x}) = \underline{\underline{\sigma}}(\underline{x})\underline{n}$$

We can also use dyadic property

$\underline{t}(\underline{n},\underline{x}) = \underline{t}(\underline{e}_j,\underline{x})n_j$ summation convention

compare

$$\left(\underline{t}(\underline{e}_j,\underline{x}) \otimes \underline{e}_j\right)\underline{n} = (\underline{e}_j \cdot \underline{n})\,\underline{t}(\underline{e}_j,\underline{x}) \qquad \underline{n} = n_i\underline{e}_i$$

$$= n_i(\underline{e}_j \cdot \underline{e}_i)\,\underline{t}(\underline{e}_j,\underline{x})$$

$$= \underline{t}(\underline{e}_j,\underline{x})n_j$$

$$\Rightarrow \underline{t}(\underline{n},\underline{x}) = \left(\underline{t}(\underline{e}_j,\underline{x}) \otimes \underline{e}_j\right)\underline{n} \qquad \underline{t} = t_i\underline{e}_i$$

$$= \left(t_i(\underline{e}_j,\underline{x})\,\underline{e}_i \otimes \underline{e}_j\right)\underline{n}$$

$$\underline{t}(\underline{n},\underline{x}) = \underline{\underline{\sigma}}(\underline{x})\,\underline{n}$$

$$\boxed{\underline{\underline{\sigma}} = \sigma_{ij}\,\underline{e}_i \otimes \underline{e}_j \quad \text{with} \quad \sigma_{ij} = t_i(\underline{e}_j,\underline{x})}$$

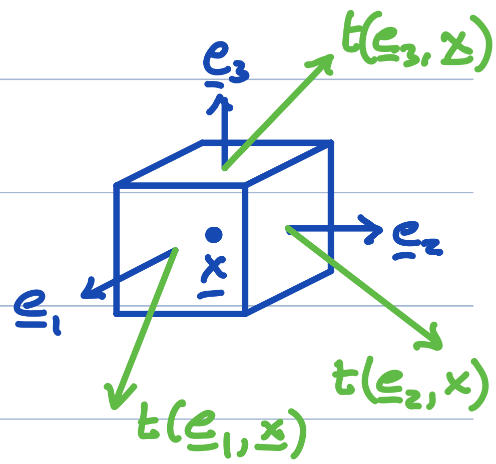
Hence σ_{ij} is the i-th component of the traction on the j-th coordinate plane.

The traction vectors on the coor. planes at $\underline{x}$ are

$$\underline{t}(\underline{e}_1, \underline{x}) = t_i(\underline{e}_1, \underline{x})\,\underline{e}_i = \sigma_{i1}(\underline{x})\,\underline{e}_i$$

$$\underline{t}(\underline{e}_2, \underline{x}) = t_i(\underline{e}_2, \underline{x})\,\underline{e}_i = \sigma_{i2}(\underline{x})\,\underline{e}_i$$

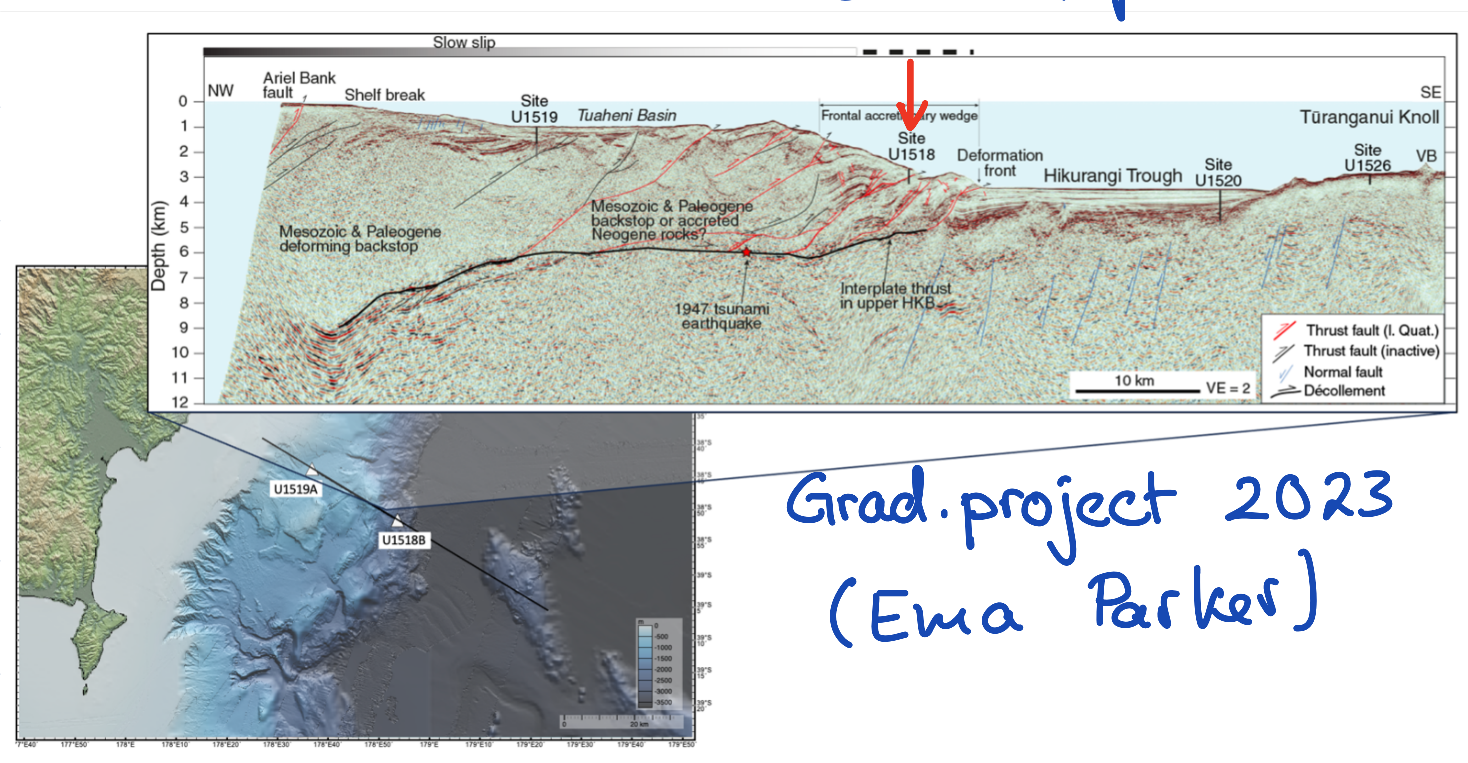
$$\underline{t}(\underline{e}_3, \underline{x}) = t_i(\underline{e}_3, \underline{x})\,\underline{e}_i = \sigma_{i3}(\underline{x})\,\underline{e}_i$$



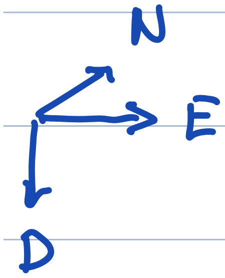
Example: North. Hikurangi subduction zone

IODP Expedition 372 & 375

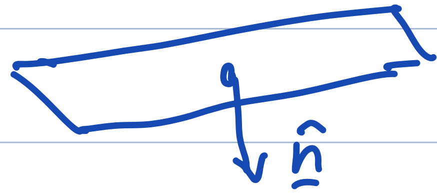
Bore hole U1518B → Papaku Fault



Grad. project 2023
(Ema Parker)



Fault normal: $\hat{\underline{n}} = \begin{array}{cc} \begin{bmatrix} 0.104 \\ -0.18 \\ -0.978 \end{bmatrix} & \begin{array}{c} N \\ E \\ D \end{array} \end{array}$



Stress on fault: $\underline{\underline{\sigma}} = \begin{bmatrix} 26 & 0.5 & 0 \\ 0.5 & 17 & 0 \\ 0 & 0 & 11 \end{bmatrix}$ MPa

traction on fault:

$$\underline{t} = \underline{\underline{\sigma}}\,\underline{n} = \begin{bmatrix} 26 & 0.5 & 0 \\ 0.5 & 17 & 0 \\ 0 & 0 & 11 \end{bmatrix} \begin{bmatrix} 0.104 \\ -0.18 \\ -0.978 \end{bmatrix} = \begin{bmatrix} 2.61 \\ -3.01 \\ -10.76 \end{bmatrix} \text{ MPa}$$

make sure you know how to evaluate traction!