Change in basis

$\underline{v}$ and $\underline{\underline{S}}$ are invariant

representations $[\underline{v}]$ and $[\underline{\underline{S}}]$ are not

Two frames $\{\underline{e}_i\}$ and $\{\underline{e}'_i\}$

Representation of $\underline{e}'_j$ in $\{\underline{e}_i\}$

$$[\underline{e}'_j] = (\underline{e}'_j \cdot \underline{e}_1)\,\underline{e}_1 + (\underline{e}'_j \cdot \underline{e}_2)\,\underline{e}_2 + (\underline{e}'_j \cdot \underline{e}_3)\,\underline{e}_3$$

$$= \underbrace{(\underline{e}'_j \cdot \underline{e}_i)}_{A_{ij}}\,\underline{e}_i$$

$$[\underline{e}'_j] = A_{ij}\,\underline{e}_i \qquad \text{where} \qquad A_{ij} = \underline{e}_i \cdot \underline{e}'_j$$

note transpose

Here $\underline{\underline{A}}$ is the change of basis tensor

$$\boxed{\underline{\underline{A}} = A_{ij}\,\underline{e}_i \otimes \underline{e}_j \qquad A_{ij} = \underline{e}_i \cdot \underline{e}'_j}$$

What kind of tensor?

Similarly we express $\underline{e}_i$ in $\{\underline{e}'_k\}$

$$[\underline{e}_i]' = (\underline{e}_i \cdot \underline{e}'_k)\,\underline{e}'_k = A_{ik}\,\underline{e}'_k$$

$$\underline{e}'_j = A_{ij}\,\underline{e}_i$$
$$\underline{e}'_j = A_{ij}(A_{ik}\,\underline{e}'_k)$$
$$= \underbrace{A_{ij}A_{ik}}_{\delta_{jk}}\,\underline{e}'_k$$
$$\underline{e}'_j = \delta_{jk}\,\underline{e}'_k$$

$$\underline{e}_i = A_{ik}\,\underline{e}'_k$$
$$\underline{e}_i = A_{ik}(A_{\ell k}\,\underline{e}_\ell)$$
$$\underline{e}_i = \underbrace{A_{ik}A_{\ell k}}_{\delta_{i\ell}}\,\underline{e}_\ell$$
$$\underline{e}_i = \delta_{i\ell}\,\underline{e}_\ell$$

$$\Rightarrow A_{ij}A_{ik} = \delta_{ik} \qquad\qquad \Rightarrow A_{ik}A_{\ell k} = \delta_{i\ell}$$

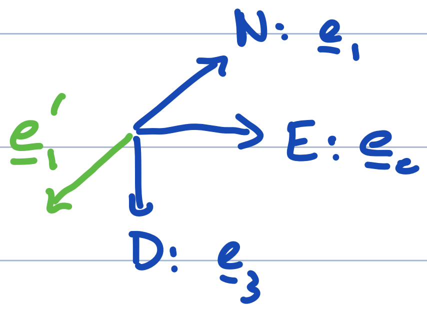
$$\boxed{\underline{\underline{A}}^T\underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{A}}^T = \underline{\underline{I}}} \Rightarrow \underline{\underline{A}} \text{ is orthonormal}$$

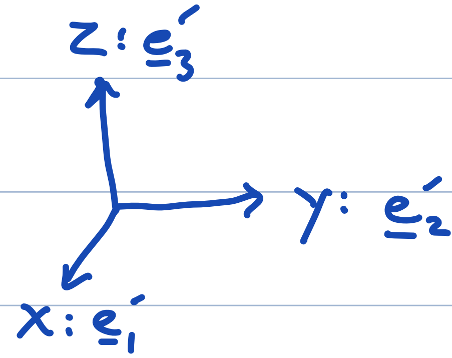
if $\{\underline{e}_i\}$ and $\{\underline{e}'_i\}$ are right handed

$$(\underline{e}_i \times \underline{e}_j)\cdot\underline{e}_k = \epsilon_{ijk} \qquad (\underline{e}'_i \times \underline{e}'_j)\cdot\underline{e}'_k = \epsilon_{ijk}$$

$\Rightarrow \underline{\underline{A}}$ must be rotation $\det(\underline{\underline{A}}) = 1$

Example: Consider NED $\{\underline{e}_i\}$ and xyz $\{\underline{e}'_i\}$





$$[\underline{e}'_1] = \underbrace{(\underline{e}_1 \cdot \underline{e}'_1)}_{-1} \underline{e}_1 + \underbrace{(\underline{e}_2 \cdot \underline{e}'_1)}_{0} \underline{e}_2 + \underbrace{(\underline{e}_3 \cdot \underline{e}_1)}_{0} \underline{e}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$[\underline{e}'_2] = [\underline{e}_2] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad [\underline{e}'_3] = -[\underline{e}_3] = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\underline{\underline{A}} = A_{ij}\, \underline{e}_i \otimes \underline{e}_j \qquad A_{ij} = \underline{e}_i \cdot \underline{e}'_j$$

$$\underline{\underline{A}} = \left([\underline{e}'_1], [\underline{e}'_2], [\underline{e}'_3]\right) = \begin{array}{c} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow \\ [\underline{e}'_1] & [\underline{e}'_2] & [\underline{e}'_3] \end{matrix} \end{array}$$

Columns of $\underline{\underline{A}}$ are representation of $\underline{e}'_i$ in $\{\underline{e}_i\}$!

Note how this is analogous to stress:

$$\underline{\underline{\sigma}} = \underbrace{\begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \underline{t}(\underline{e}_1) & \underline{t}(\underline{e}_2) & \underline{t}(\underline{e}_3) \\ \downarrow & \downarrow & \downarrow \end{bmatrix}}_{\text{tractions on coord. planes}}$$

traction on any plane:

$$\underline{t}_n = \underline{\underline{\sigma}}\,\underline{n} = \underbrace{n_1 \underline{t}(\underline{e}_i) + n_2 \underline{t}(\underline{e}_2) + n_3 \underline{t}(\underline{e}_3)}_{\text{linear combination.}}$$

Similarly:

$$\underline{\underline{A}} = \underbrace{\begin{bmatrix} | & | & | \\ [\underline{e}_1'] & [\underline{e}_2'] & [\underline{e}_3'] \\ | & | & | \end{bmatrix}}_{\text{representation of base vectors}}$$

representation of any vector.

$$\underline{v} = \underline{\underline{A}}\,\underline{v}' = \underbrace{v_1' [\underline{e}_1'] + v_2' [\underline{e}_2'] + v_3' [\underline{e}_3']}_{\text{linear combination}}$$

Change in representation of vectors

$[\underline{v}]$ and $[\underline{\underline{S}}]$ in $\{\underline{e}_i\}$

$[\underline{v}]'$ and $[\underline{\underline{S}}]'$ in $\{\underline{e}_i'\}$

$$\boxed{[\underline{v}] = [\underline{\underline{A}}][\underline{v}]' \quad \text{and} \quad [\underline{v}]' = [\underline{\underline{A}}]^T[\underline{v}]}$$

to see this $\quad \underline{v} = v_i \underline{e}_i = v_j' \underline{e}_j' \quad$ but $\underline{e}_j' = A_{ij}\underline{e}_i$

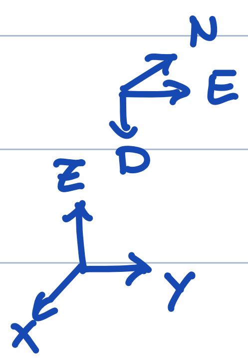
$$= v_j'(A_{ij}\underline{e}_i)$$

$$v_i \underline{e}_i = A_{ij} v_j' \underline{e}_i$$

by comparison: $\quad v_i = A_{ij} v_j'$ ✓

Example: NED $\{\underline{e}_i\}$

XYZ $\{\underline{e}_i'\}$



HW1: $[\underline{a}]' = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad [\underline{a}] = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix}$ and $[\underline{\underline{A}}] = \begin{bmatrix} -1 & & \\ & 1 & \\ & & -1 \end{bmatrix}$

$$[\underline{\underline{A}}][\underline{a}]' = \begin{bmatrix} -1 & & \\ & 1 & \\ & & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix} = [\underline{a}] \quad ✓$$

Change in representation for tensors

$$\boxed{\begin{aligned} [\underline{\underline{S}}] &= [\underline{\underline{A}}][\underline{\underline{S}}]'[\underline{\underline{A}}]^T \\ [\underline{\underline{S}}]' &= [\underline{\underline{A}}]^T[\underline{\underline{S}}][\underline{\underline{A}}] \end{aligned}}$$

$\Rightarrow$ HW 4

with $\underline{\underline{A}} = A_{ij}\, e_i \otimes e_j' \qquad A_{ij} = e_i \cdot e_j'$

$\underline{\underline{A}}^T\underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{A}}^T = \underline{\underline{I}} \qquad \det(\underline{\underline{A}}) = 1 \quad$ (rotation)

$\underline{\underline{A}}^T = \underline{\underline{A}}^{-1}$

Look familiar? $\quad [\underline{\underline{S}}]' = [\underline{\underline{A}}]^{-1}[\underline{\underline{S}}][\underline{\underline{A}}]$

Diagonalizing a matrix: $\underline{\underline{\Lambda}} = \underline{\underline{V}}^{-1}\underline{\underline{A}}\,\underline{\underline{V}}$

$$\underline{\underline{\Lambda}} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} \qquad \underline{\underline{V}} = \begin{bmatrix} | & | & | \\ \underline{v}_1 & \underline{v}_2 & \underline{v}_3 \\ | & | & | \end{bmatrix}$$

$$(\underline{\underline{A}} - \lambda_p \underline{\underline{I}})\,\underline{v}_p = 0$$

$\Rightarrow$ eigenvalues & eigenvectors

Eigenvalues & vectors of tensors

By the eigen pair of $\underline{\underline{S}} \in \mathcal{V}^2$ we mean the scalar λ and the vector $\underline{v}$ such that

$$\boxed{\underline{\underline{S}}\,\underline{v} = \lambda\,\underline{v}}$$

λ = eigenvalue $\qquad$ $\underline{v}$ = eigenvector

λ_p's are roots of characteristic polynomial

$$p(\lambda) = \det(\underline{\underline{S}} - \lambda \underline{\underline{I}}) = 0$$

For each λ_p we one or more $\underline{v}_p$'s satisfying

$$(\underline{\underline{S}} - \lambda_p \underline{\underline{I}})\,\underline{v}_p = \underline{0}$$

In continuum mechanics we are mostly concerned with symmetric tensors, eg. stress. $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$

Eigen problem for symmetric tensors

1) All λ_p's real
2) All λ_p's are positive ($\underline{\underline{S}}$ sym. pos. def.)
3) All $\underline{v}_p$'s corresponding to distinct λ_p's are orthogonal

$\underline{\underline{S}}$ is symmetric positive definite (spd)
if $\quad \underline{v} \cdot \underline{\underline{S}}\, \underline{v} > 0 \quad$ for all $\underline{v} \in \mathcal{V}$

by def. of eigenpair $\quad \underline{\underline{S}}\,\underline{v} = \lambda \underline{v}$

$$\underline{v} \cdot \lambda \underline{v} = \lambda |\underline{v}|^2 > 0 \rightarrow \lambda > 0$$

For orthogonality consider two eigenpairs $(\lambda, \underline{v})$ and $(\omega, \underline{u})$ so that $\lambda \neq \omega$

$$\underline{\underline{S}}\,\underline{v} = \lambda \underline{v} \quad \text{and} \quad \underline{\underline{S}}\,\underline{u} = \omega \underline{u}$$

Consider $\quad \lambda(\underline{v} \cdot \underline{u}) = \underline{\underline{S}}\,\underline{v} \cdot \underline{u} = \underline{v} \cdot \underline{\underline{S}}^T \underline{u} \quad (\underline{\underline{S}}^T = \underline{\underline{S}})$

$$= \underline{v} \cdot \underline{\underline{S}}\,\underline{u} = \omega(\underline{v} \cdot \underline{u})$$

$$\lambda(\underline{v} \cdot \underline{u}) = \omega(\underline{v} \cdot \underline{u})$$

since $\lambda \neq \omega \Rightarrow \underline{v} \cdot \underline{u} = \underline{0}$

Spectral decomposition

If $\underline{\underline{S}} = \underline{\underline{S}}^T$ we can choose the set of eigen-vectors as frame $\{\underline{e}'\}$ so that

$$\boxed{[\underline{\underline{S}}]' = \sum_{i=1}^{3} \lambda_i (\underline{e}'_i \otimes \underline{e}'_i)} \qquad \underline{e}'_i = \underline{v}_i$$

where

$$[\underline{\underline{S}}]' = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad \text{diagonal tensor}$$

To see this consider $[\underline{\underline{S}}]'[\underline{v}_i]' = \lambda_p [\underline{v}_i]'$ where $[\underline{v}_p]' = \underline{e}'_i$

In frame $\{\underline{e}'_i\}$ we have $[\underline{\underline{I}}]' = \underline{e}'_i \otimes \underline{e}'_i$

$$[\underline{\underline{S}}]'[\underline{\underline{I}}]' = [\underline{\underline{S}}]'(\underline{e}'_i \otimes \underline{e}'_i) = ([\underline{\underline{S}}]'\underline{e}'_i) \otimes \underline{e}'_i \quad \text{use } \underline{\underline{S}}\,\underline{e}'_i = \lambda_i \underline{e}'_i$$

$$= \sum_{i=1}^{3} ([\underline{\underline{S}}]'\underline{e}'_i) \otimes \underline{e}'_i = \sum_{i=1}^{3} (\lambda_i \underline{e}'_i) \otimes \underline{e}'_i$$

$$= \sum_{i=1}^{3} \lambda_i (\underline{e}'_i \otimes \underline{e}'_i)$$

where we have used associative properties of dyadic product: $\underline{\underline{A}}(\underline{a} \otimes \underline{b}) = (\underline{\underline{A}}\,\underline{a}) \otimes \underline{b}$ $\Rightarrow$ HW3

$$\alpha(\underline{a} \otimes \underline{b}) = (\alpha \underline{a}) \otimes \underline{b}$$

Change of basis $\{\underline{e}_i\}$ to $\{\underline{e}_i'\}$

principal frame

$$\underline{\underline{A}} = (\underline{e}_i \cdot \underline{e}_j')\ \underline{e}_i \otimes \underline{e}_j$$

$$= \Big[[\underline{e}_1'] \, [\underline{e}_2'] \, [\underline{e}_3'] \Big] = \Big[[\underline{v}_1] \, [\underline{v}_2] \, [\underline{v}_3] \Big] = \underline{\underline{V}}$$

representation of eigenvectors $\underline{\underline{S}}\,\underline{v}_p = \lambda_p \underline{v}_p$ in $\{\underline{e}_i\}$

$$\Rightarrow \quad [\underline{\underline{S}}] = [\underline{\underline{V}}]\,[\underline{\underline{S}}]'\,[\underline{\underline{V}}]^T$$

relation to diagonalization of symmetric matrices:

$$[\underline{\underline{S}}]' = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} = \underline{\underline{\Lambda}}$$

$[\underline{\underline{V}}]^T = [\underline{\underline{V}}]^{-1}$ orthogonal

in matrices $[\underline{\underline{S}}] = \underline{\underline{S}}$ always in $\{\underline{e}_i\}$

$$\underline{\underline{S}} = \underline{\underline{V}}\,\underline{\underline{\Lambda}}\,\underline{\underline{V}}^{-1}$$

$$\boxed{\underline{\underline{\Lambda}} = \underline{\underline{V}}^{-1}\,\underline{\underline{S}}\,\underline{\underline{V}}}$$

diagonal   full

$\underline{\underline{S}} = \underline{\underline{S}}^T$

$\underline{\underline{V}}$ = eigenvector matrix

$\underline{\underline{\Lambda}}$ = diagonal eigenvalue mat.

$[\underline{\underline{\sigma}}]$ representation in $\{\underline{e}_i\}$

$\Rightarrow$ eigenvectors tensor = change of basis

$$\underline{\underline{V}} = \Big[[\underline{v}_1] \, [\underline{v}_2] \, [\underline{v}_3] \Big] \equiv \underline{\underline{A}}$$

$$\{\underline{e}_i\} \longleftrightarrow \{\underline{e}_i'\} = \{\underline{v}_i\}$$

change of basis between $\{\underline{e}_i\}$ and $\{\underline{e}_i'\}$

$\{\underline{e}_i'\} = \{\underline{v}_i\}$ principal frame/eigenframe

To find representation of stress in NED

$\Rightarrow$ represent eigenvectors in NED to form $\underline{\underline{A}}$