Constitutive Theory

Common constitutive laws:

Newtonian fluid: $\underline{\underline{\sigma}} = -p\underline{\underline{I}} + \lambda(\nabla\cdot\underline{v})\underline{\underline{I}} + \eta(\nabla\underline{v} + \nabla\underline{v}^T)$

$\Rightarrow$ $\underline{v}$ = velocity & p = thermo. pressure

$\lambda = 2^{nd}$ viscosity η = dyn. viscosity

Linear elastic solid: $\underline{\underline{\sigma}} = \lambda\nabla\cdot\underline{u}\,\underline{\underline{I}} + \mu(\nabla\underline{u} + \nabla\underline{u}^T)$

$\Rightarrow$ $\underline{u}$ = displacement

λ, μ = lame parameters

Both derive from the functional form

$$\boxed{\underline{\underline{\sigma}}(\underline{\underline{E}}) = \lambda\,\mathrm{tr}(\underline{\underline{E}})\underline{\underline{I}} + 2\mu\,\mathrm{sym}(\underline{\underline{E}})}$$

Newtonian fluid: $\underline{\underline{E}} = \nabla\underline{v}$

Linear elastic solid: $\underline{\underline{E}} = \nabla\underline{u}$

remember $\nabla\cdot\underline{a} = \mathrm{tr}(\nabla\underline{a})$

$\Rightarrow$ direct for lin. elastic solid

Note: fluid has additional term due to pressure

Why do constitutive relations have this form?

Change of observer

In Lecture 6 we discussed Change in basis

$$\underline{v} = \underline{\underline{A}}\,\underline{v}' \quad \text{and} \quad \underline{\underline{S}} = \underline{\underline{A}}\,\underline{\underline{S}}'\underline{\underline{A}}^T$$

where $\underline{\underline{A}}$ is change in basis tensor.

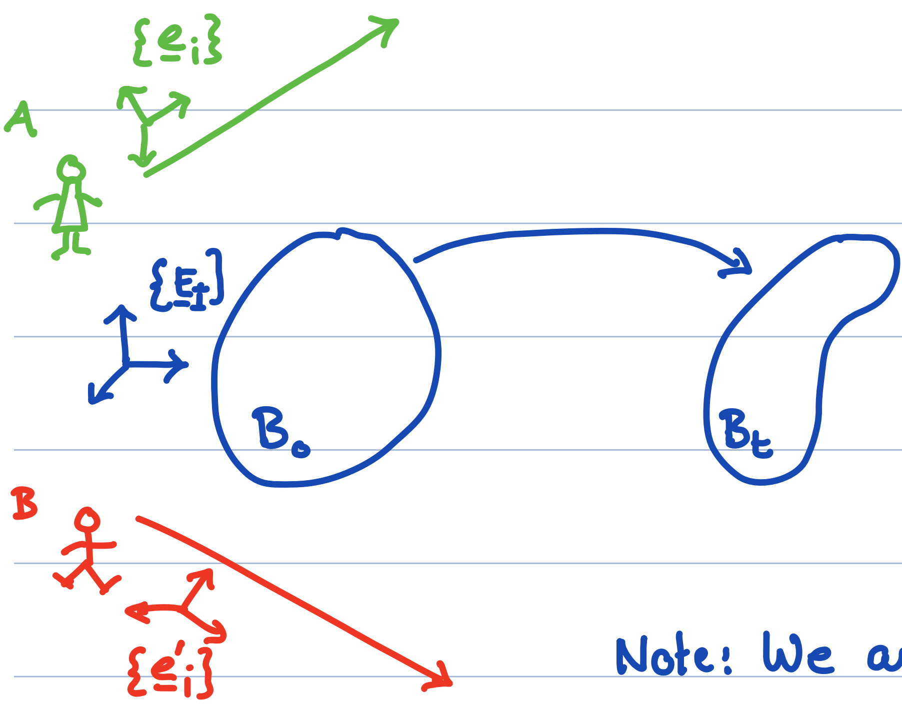
$\Rightarrow$ $\underline{\underline{A}}$ is a rotation: $\underline{\underline{A}} = \underline{\underline{Q}}$

Q is a rotation: 1) orthonormal $\underline{\underline{Q}}\,\underline{\underline{Q}}^T = \underline{\underline{Q}}^T\underline{\underline{Q}} = \underline{\underline{I}}$

2) $\det(\underline{\underline{Q}}) = 1$

Change in basis is passive change of frame.

Active change in frame $\rightarrow$ change in observer



$$\underline{x} = \varphi(\underline{X}, t)$$

$$\underline{x}' = \varphi'(\underline{X}, t)$$

Note: Material ref. frame is comon

Note: We assume both observers use same clock.

Since change in observer cannot induce a deformation. Two ref. frames must be related by a rigid body motion.

$$\boxed{\underline{x}' = Q(t)\,\varphi(\underline{X},t) + \underline{c}(t)} \qquad \text{Eulerian transformation}$$

$\underline{\underline{Q}}$ = rotation $\qquad$ $\underline{c}$ = translation

Our description of forces and deformations cannot depend on the observer (objective).

Effect on kinematic quantities

$$\underline{x} = \varphi(\underline{X},t) \qquad \nabla\varphi = \underline{\underline{F}}$$

$$\underline{x}' = \varphi'(\underline{X},t) = Q\,\varphi(\underline{X},t) + c \qquad \underline{\nabla\varphi' = \underline{\underline{Q}}\,\underline{\underline{F}} = \underline{\underline{F}}'}$$

Right Cauchy-Green Strain tensor

$$\underline{\underline{C}}' = \underline{\underline{F}}'^T\,\underline{\underline{F}}' = (\underline{\underline{Q}}\,\underline{\underline{F}})^T(\underline{\underline{Q}}\,\underline{\underline{F}}) = \underline{\underline{F}}^T\underline{\underline{Q}}^T\underline{\underline{Q}}\,\underline{\underline{F}} = \underline{\underline{F}}^T\underline{\underline{F}} = \underline{\underline{C}}$$

$\Rightarrow$ not affected by rigid body motion because it is a material tensor $C = C(\underline{X})$ (naturally objective)

What about <u>spatial</u> tensors? $\nabla\underline{u}$, $\underline{\underline{\varepsilon}}$, $\nabla\underline{v}$, $\underline{\underline{d}}$

Axiom of frame indifference

Spatial fields ϕ, $\underline{w}$ and $\underline{\underline{S}}$ are frame indifferent or objective if for all superposed rigid body motions $\underline{x}' = \underline{\underline{Q}}\,\underline{x} + \underline{c}$ we have

$$\boxed{\begin{aligned} \phi'(\underline{x}', t) &= \phi(\underline{x}, t) \\ \underline{w}'(\underline{x}', t) &= \underline{\underline{Q}}\,\underline{w}(\underline{x}, t) \\ \underline{\underline{S}}'(\underline{x}', t) &= \underline{\underline{Q}}\,\underline{\underline{S}}(\underline{x}, t)\,\underline{\underline{Q}}^T \end{aligned}}$$

scalar field

vector field

tensor field

$\Rightarrow$ from Lecture 6.

Is spatial velocity gradient objective?

From lecture 16: $\underline{\underline{l}} = \nabla_x \underline{v} = \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}$

$\underline{\underline{F}}' = \underline{\underline{Q}}\,\underline{\underline{F}}$ $\qquad$ $\underline{\underline{l}}' = \nabla_{x'} \underline{v}' = \dot{\underline{\underline{F}}}'\,\underline{\underline{F}}'^{-1}$

$$\dot{\underline{\underline{F}}}' = \frac{d}{dt}(\underline{\underline{Q}}\,\underline{\underline{F}}) = \underline{\underline{Q}}\,\dot{\underline{\underline{F}}} + \dot{\underline{\underline{Q}}}\,\underline{\underline{F}}$$

$$\underline{\underline{F}}'^{-1} = (\underline{\underline{Q}}\,\underline{\underline{F}})^{-1} = \underline{\underline{F}}^{-1}\underline{\underline{Q}}^{-1} = \underline{\underline{F}}^{-1}\underline{\underline{Q}}^T$$

$$\begin{aligned} \underline{\underline{l}}' = \dot{\underline{\underline{F}}}'\,\underline{\underline{F}}'^{-1} &= (\underline{\underline{Q}}\,\dot{\underline{\underline{F}}} + \dot{\underline{\underline{Q}}}\,\underline{\underline{F}})\,\underline{\underline{F}}^{-1}\underline{\underline{Q}}^T \\ &= \underline{\underline{Q}}\,\dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}\underline{\underline{Q}}^T + \dot{\underline{\underline{Q}}}\,\underline{\underline{F}}\,\underline{\underline{F}}^{-1}\underline{\underline{Q}}^T \\ &= \underline{\underline{Q}}\,\underline{\underline{l}}\,\underline{\underline{Q}}^T + \dot{\underline{\underline{Q}}}\,\underline{\underline{Q}}^T \end{aligned}$$

$\Rightarrow \underline{\underline{\ell}}' = \underline{\underline{Q}}\,\underline{\underline{\ell}}\,\underline{\underline{Q}}^T + \dot{\underline{\underline{Q}}}\,\underline{\underline{Q}}^T$ <u>not</u> objective!

$\nabla_x \underline{v}$ is not used in constitutive laws

The "non-objective" term is $\underline{\underline{\Omega}} = \dot{\underline{\underline{Q}}}\,\underline{\underline{Q}}^T$

it represents rigid body angular velocity between observers. (HW9)

rate of rotation: $\underline{\underline{\Omega}} = -\underline{\underline{\Omega}}^T$ skew-symmetric (HW9)

Non-objetive part of $\underline{\underline{\ell}} = \nabla_x \underline{v}$ is skew-sym.

$\Rightarrow$ simply take symmetric part of $\underline{\underline{\ell}}$!

$\underline{\underline{d}} = \mathrm{sym}(\underline{\underline{\ell}}) = \frac{1}{2}(\nabla_x \underline{v} + \nabla_x \underline{v}^T)$

rate of strain tensor is objective

$\Rightarrow$ used in constitutive laws

Frame indifferent/isotropic functions

Not just tensors themselves, but the functions relating them must be frame indifferent

$$\boxed{\underline{\underline{\sigma}} = \hat{\underline{\underline{\sigma}}}(\underline{\underline{d}})}$$

stress tensor — constitutive function — rate of strain tensor

$\hat{\underline{\underline{\sigma}}}$ is not directly dependent on frame, but input fields are

Two frames $\{\underline{e}_i\}$ and $\{\underline{e}_i'\}$ then frame indifference requires:

1) $\underline{\underline{\sigma}}' = \hat{\underline{\underline{\sigma}}}(\underline{\underline{d}}') = \hat{\underline{\underline{\sigma}}}(\underline{\underline{Q}}\,\underline{\underline{d}}\,\underline{\underline{Q}}^T)$

2) $\underline{\underline{\sigma}}' = \underline{\underline{Q}}\,\underline{\underline{\sigma}}\,\underline{\underline{Q}}^T = \underline{\underline{Q}}\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{d}})\,\underline{\underline{Q}}^T$

$\Rightarrow \hat{\underline{\underline{\sigma}}}(\underline{\underline{Q}}\,\underline{\underline{d}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{d}})\,\underline{\underline{Q}}^T$

More general: $\boxed{\underline{\underline{G}}(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\underline{\underline{G}}(\underline{\underline{S}})\,\underline{\underline{Q}}^T}$

$\underline{\underline{Q}}$ = change in basis/rotation

$\underline{\underline{S}}$ = tensor $\qquad \underline{\underline{G}}$ = tensor valued tensor function

Representation Theorem

Most general isotropic function $\underline{\underline{G}}(\underline{\underline{A}})$ that maps symmetric tensors to symmetric tensors is

$$\boxed{\underline{\underline{G}}(\underline{\underline{S}}) = \alpha_0(I_s)\,\underline{\underline{I}} + \alpha_1(I_s)\,\underline{\underline{S}} + \alpha_2(I_s)\,\underline{\underline{S}}^2}$$

Rivlin - Ericksen Rep. Thm.

here α_0, α_1 and α_2 are scalar funtions of the principle invariants of $\underline{\underline{S}}$ → later

Check $\underline{\underline{G}}$ is isotropic/frame invariant

assume α_0, α_1, α_2 are constants.

$$\underline{\underline{G}}(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \alpha_0 \underset{\overset{\uparrow}{\underline{\underline{Q}}\,\underline{\underline{I}}\,\underline{\underline{Q}}^T}}{\underline{\underline{I}}} + \alpha_1 \underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T + \alpha_2\, \underline{\underline{Q}}\,\underline{\underline{S}}\,\overset{\underline{\underline{I}}}{\cancel{\underline{\underline{Q}}^T \underline{\underline{Q}}}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T$$

$$= \underline{\underline{Q}}\left(\alpha_0 \underline{\underline{I}} + \alpha_1 \underline{\underline{S}} + \alpha_2 \underline{\underline{S}}^2\right)\underline{\underline{Q}}^T$$

$$= \underline{\underline{Q}}\,\underline{\underline{G}}(\underline{\underline{S}})\,\underline{\underline{Q}}^T \quad ✓$$

⇒ if $\alpha_2 = 0$ linear isotropic representation Thm

Gurtin (1974) "A short proof ..."

Principal Invariants of Symmetric Tensors

Both stress and strain are symmetric

$$(\underline{\underline{S}} - \lambda_p \underline{\underline{I}})\, \underline{v}_p = \underline{0}$$

$\Rightarrow$ 1) All λ_p's are real

2) All λ_p's are positive (sym. pos. def.)

3) All $\underline{v}_p$'s corresponding to distinct λ_p's are orthogonal

Spectral decomposition: $\underline{\underline{S}} = \sum_{p=1}^{3} \lambda_p \, \underline{e}_p \otimes \underline{e}_p$

Principal invariants of $\underline{\underline{S}}$

$$\boxed{\begin{aligned} I_1(\underline{\underline{S}}) &= \mathrm{tr}(\underline{\underline{S}}) = \lambda_1 + \lambda_2 + \lambda_3 \\ I_2(\underline{\underline{S}}) &= \tfrac{1}{2}\left[\mathrm{tr}(\underline{\underline{S}})^2 - \mathrm{tr}(\underline{\underline{S}}^2)\right] = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 \\ I_3(\underline{\underline{S}}) &= \det(\underline{\underline{S}}) = \lambda_1\lambda_2\lambda_3 \end{aligned}}$$

3 scalars are frame indifferent.

$\Rightarrow$ usefull in constitutive functions.

$I_S = \{ I_i(\underline{\underline{S}}) \}$ set of invariants of $\underline{\underline{S}}$

$\{\underline{e}_i\}$ and $\{\underline{e}'_i\}$

1) Invariance of trace = 1^{st} invariant

$tr(\underline{\underline{S}}) = tr(\underline{\underline{S}}')$ where $\underline{\underline{S}} = \underline{\underline{Q}}\,\underline{\underline{S}}'\,\underline{\underline{Q}}^T$

$S_{ij} = Q_{ik} S'_{kl} Q_{jl}$

$$tr(\underline{\underline{S}}) = S_{ii} = Q_{ik} S'_{kl} Q_{il}$$
$$= S'_{kl} \underbrace{Q_{ik} Q_{il}}_{\delta_{kl}} = S'_{kk} = tr(\underline{\underline{S}}') \checkmark$$

2) Invariance of determinant = 3^{rd} invariant

$det(\underline{\underline{S}}) = det(\underline{\underline{S}}')$ $\underline{\underline{S}} = \underline{\underline{Q}}\,\underline{\underline{S}}'\,\underline{\underline{Q}}^T$

$$det(\underline{\underline{Q}}\,\underline{\underline{S}}'\,\underline{\underline{Q}}^T) = \cancel{det(\underline{\underline{Q}})}^{1}\, det(\underline{\underline{S}}')\, \cancel{det(\underline{\underline{Q}}^T)}^{1}$$
$$= det(\underline{\underline{S}}') \checkmark$$

Invariance of 2^{nd} follows from 1^{st} !

$\Rightarrow$ back to representation theorem

$$\underline{\underline{G}}(\underline{\underline{S}}) = \alpha_0(I_s)\,\underline{\underline{I}} + \alpha_1(I_s)\,\underline{\underline{S}} + \alpha_2(I_s)\,\underline{\underline{S}}^2$$

$\Rightarrow$ $\underline{\underline{G}}$ remains frame indifferent if

α_i's are functions of invariants of $\underline{\underline{S}}$

Linear isotropic constitutive function

$\underline{\underline{G}}(\underline{\underline{S}})$ is linear in $\underline{\underline{S}}$

$$\Rightarrow \quad \alpha_0(I_S) = c_0 \, \mathrm{tr}(\underline{\underline{S}}) + c_1$$

$$\alpha_1(I_S) = c_2$$

$$\alpha_2(I_S) = 0$$

if $\underline{\underline{G}}(\underline{\underline{0}}) = \underline{\underline{0}} \quad \Rightarrow \quad c_1 = 0$

with $c_0 = \lambda$ and $c_2 = 2\mu$

$$\Rightarrow \quad \boxed{\underline{\underline{G}}(\underline{\underline{S}}) = \lambda \, \mathrm{tr}(\underline{\underline{S}}) \, \underline{\underline{I}} + 2\mu \, \underline{\underline{S}}} \qquad \underline{\underline{S}} = \underline{\underline{S}}^T$$

or for non-symmetric $\underline{\underline{E}}$

$$\boxed{\underline{\underline{G}}(\underline{\underline{E}}) = \lambda \, \mathrm{tr}(\underline{\underline{E}}) \, \underline{\underline{I}} + 2\mu \, \mathrm{sym}(\underline{\underline{E}})}$$

choosing $\underline{\underline{E}} = \nabla \underline{u} \;\Rightarrow$ linear elastic stress function

$$\underline{\underline{\sigma}}(\underline{x}, t) = \hat{\underline{\underline{\sigma}}}(\nabla \underline{u}) = \lambda (\nabla \cdot \underline{u}) \, \underline{\underline{I}} + \mu (\nabla \underline{u} + \nabla^T \underline{u})$$

where $\nabla \cdot \underline{u} = \mathrm{tr}(\nabla \underline{u})$!

Linear elastic solid: $\boxed{\underline{\underline{\sigma}} = \lambda (\nabla \cdot \underline{v}) \, \underline{\underline{I}} + \mu (\nabla \underline{u} + \nabla^T u)}$

choosing $\underline{\underline{E}} = \nabla \underline{v} \Rightarrow$ fluid ?

$$\underline{\underline{\sigma}}(\underline{x},t) = \hat{\underline{\underline{\sigma}}}(\nabla \underline{v}) = \lambda \nabla \cdot \underline{v} + \mu (\nabla \underline{v} + \nabla^T \underline{v})$$

In hydrostatic fluid: $\underline{v} = \underline{0}$ and $\underline{\underline{\sigma}} = -p\underline{\underline{I}}$

$$\hat{\underline{\underline{\sigma}}}(\underline{\underline{0}}) = \underline{\underline{0}} \neq -p\underline{\underline{I}} \quad \Rightarrow \quad c_1 = -p$$

but $p(\underline{x},t)$ is scalar function $\rightarrow$ invariant

Newtonian fluid: $\boxed{\underline{\underline{\sigma}} = -p\underline{\underline{I}} + \lambda(\nabla \cdot \underline{v})\underline{\underline{I}} + \mu(\nabla \underline{v} + \nabla^T \underline{v})}$