Derivatives of tensor functions

So far we have considered fields: $\phi(\underline{x}), \underline{v}(\underline{x}), \underline{\underline{S}}(\underline{x})$

Now we are interested in tensor functions

- scalar-valued tensor functions: $\psi = \psi(\underline{\underline{S}})$
- tensor-valued tensor functions: $\underline{\underline{\Sigma}} = \underline{\underline{\Sigma}}(\underline{\underline{S}})$

Derivatives of scalar-valued tensor functions

Typical examples: $\det \underline{\underline{A}}$ or $\operatorname{tr} \underline{\underline{A}}$

Definition: A function $\psi(\underline{\underline{S}})$ is differentiable at $\underline{\underline{A}}$ if there exists a tensor $D\psi(\underline{\underline{A}})$, s.t.

$$\psi(\underline{\underline{A}} + \underline{\underline{H}}) = \psi(\underline{\underline{A}}) + D\psi(\underline{\underline{A}}) : \underline{\underline{H}} + o(|\underline{\underline{H}}|)$$

or equivalently with $\underline{\underline{H}} = \epsilon \underline{\underline{U}}$

$$\boxed{D\psi(\underline{\underline{A}}) : \underline{\underline{U}} = \frac{d}{d\epsilon} \psi(\underline{\underline{A}} + \epsilon \underline{\underline{U}})\Big|_{\epsilon=0}} \qquad \text{for all } \underline{\underline{U}} \in \mathcal{V}^2$$

$D\psi(\underline{\underline{A}})$ is called the derivative of ψ at $\underline{\underline{A}}$

In frame $\{\underline{e}_i\}$ we have $\boxed{D\psi(\underline{\underline{A}}) = \frac{\partial \psi}{\partial A_{ij}} \underline{e}_i \otimes \underline{e}_j}$

To see this write $\psi(A_{11}, A_{12}, \dots, A_{33})$

and $\underline{\underline{U}} = U_{kl}\, \underline{e}_k \otimes \underline{e}_l$ then

$$\psi(\bar{\underline{\underline{A}}} + \epsilon \underline{\underline{U}}) = \psi(\underbrace{\bar{A}_{11} + \epsilon U_{11}}_{A_{11}}, \underbrace{\bar{A}_{12} + \epsilon U_{12}}_{A_{12}}, \dots, \bar{A}_{33} + \epsilon U_{33})$$

by chain rule

$$\underline{D\psi(\underline{\underline{A}})} : \underline{\underline{U}} = \frac{d}{d\varepsilon} \psi(\bar{A}_{11} + \epsilon U_{11}, \dots, \bar{A}_{33} + \epsilon U_{33}) \Big|_{\epsilon = 0}$$

$$= \frac{\partial \psi}{\partial A_{11}} U_{11} + \frac{\partial \psi}{\partial A_{12}} U_{12} + \dots + \frac{\partial \psi}{\partial A_{33}} U_{33} = \frac{\partial \psi}{\partial A_{ij}} U_{ij}$$

$$= \left(\underline{\frac{\partial \psi}{\partial A_{ij}} \underline{e}_i \otimes \underline{e}_j} \right) : \left(U_{kl}\, \underline{e}_k \otimes \underline{e}_l \right)$$

result is implied by the arbitraryness of $\underline{\underline{U}}$

Derivative of trace

$\psi(\underline{\underline{A}}) = \mathrm{tr}(\underline{\underline{A}}) = A_{ii}$ Using the definition

$$D\mathrm{tr}(\underline{\underline{A}}) = \frac{\partial A_{ii}}{\partial A_{kl}} \underline{e}_k \otimes \underline{e}_l = \delta_{ik} \delta_{il}\, \underline{e}_k \otimes \underline{e}_l = \underline{e}_i \otimes \underline{e}_i = \underline{\underline{I}}$$

$$\boxed{D\mathrm{tr}(\underline{\underline{A}}) = \underline{\underline{I}}}$$

Derivative of determinant

Let $\psi(\underline{\underline{A}}) = \det(\underline{\underline{A}})$, if $\underline{\underline{A}}$ is invertible

$$\boxed{D\det(\underline{\underline{A}}) = \det(\underline{\underline{A}})\,\underline{\underline{A}}^{-T}}$$

Note this takes some work !

Start by using the directional derivative

$$D\det(\underline{\underline{A}} + \epsilon\underline{\underline{U}}) = \frac{d}{d\epsilon}\det(\underline{\underline{A}} + \epsilon\underline{\underline{U}})\Big|_{\epsilon=0}$$

First simplify expansion

$$\det(\epsilon\underline{\underline{U}} + \underline{\underline{A}}) = \det\left(\epsilon\underline{\underline{A}}(\underline{\underline{A}}^{-1}\underline{\underline{U}} + \tfrac{1}{\epsilon}\underline{\underline{I}})\right) \qquad \frac{1}{\epsilon} = -\lambda$$

$$= \det(\epsilon\underline{\underline{A}})\det(\underline{\underline{A}}^{-1}\underline{\underline{U}} - \lambda\underline{\underline{I}})$$

$$= \epsilon^3\det(\underline{\underline{A}})\det(\underline{\underline{A}}^{-1}\underline{\underline{U}} - \lambda\underline{\underline{I}})$$

from definition of principal invariants

$$\det(\underline{\underline{A}}^{-1}\underline{\underline{U}} - \lambda\underline{\underline{I}}) = -\lambda^3 + \lambda^2 I_1(\underline{\underline{A}}^{-1}\underline{\underline{U}}) - \lambda I_2(\underline{\underline{A}}^{-1}\underline{\underline{B}}) + I_3(\underline{\underline{A}}^{-1}\underline{\underline{B}})$$

$$= -\left(-\frac{1}{\epsilon}\right)^3 + \left(-\frac{1}{\epsilon}\right)^2 I_1 + \frac{1}{\epsilon}I_2 + I_3$$

$$= \frac{1}{\epsilon^3} + \frac{1}{\epsilon^2}I_1 + \frac{1}{\epsilon}I_2 + I_3$$

substituting above $\Rightarrow$ expansion in ϵ

$$\det(\underline{\underline{A}} + \epsilon \underline{\underline{U}}) = \epsilon^3 \det(\underline{\underline{A}}) \left(\frac{1}{\epsilon^3} + \frac{1}{\epsilon^2} I_1 + \frac{1}{\epsilon} I_2 + I_3 \right)$$

$$= \det(\underline{\underline{A}}) \left(1 + \epsilon I_1 + \epsilon^2 I_2 + \epsilon^3 I_3 \right)$$

substitute into directional derivative

$$D\det(\underline{\underline{A}}) : \underline{\underline{U}} = \frac{d}{d\epsilon} \det(\underline{\underline{A}} + \epsilon \underline{\underline{U}}) \Big|_{\epsilon=0} = \det(\underline{\underline{A}}) I_1(\underline{\underline{A}}^{-1} \underline{\underline{U}})$$

since $I_1(\underline{\underline{A}}^{-1}\underline{\underline{U}}) = \mathrm{tr}(\underline{\underline{A}}^{-1}\underline{\underline{U}}) = \mathrm{tr}(A^{-1}_{ij} U_{jk} \underline{e}_i \otimes \underline{e}_k)$

$$= A^{-1}_{ij} U_{ji} = A^{-T}_{ji} U_{ji} = \underline{\underline{A}}^{-T} : \underline{\underline{U}}$$

so that

$$\underline{D\det(\underline{\underline{A}})} : \underline{\underline{U}} = \underline{\det(\underline{\underline{A}})\, \underline{\underline{A}}^{-T}} : \underline{\underline{U}}$$

the result follows from the arbitraryness of $\underline{\underline{U}}$

Time derivative of scalar valued tensor function

Let $\underline{\underline{S}} = \underline{\underline{S}}(t) \in \mathcal{V}^2$. In stationary frame $\{\underline{e}_i\}$

$\underline{\underline{S}}(t) = S_{ij}(t)\, \underline{e}_i \otimes \underline{e}_j$ so that

$$\boxed{\dot{\underline{\underline{S}}} = \frac{d\underline{\underline{S}}}{dt} = \frac{dS_{ij}}{dt}\, \underline{e}_i \otimes \underline{e}_j}$$

How do we compute $\frac{d}{dt}\psi(\underline{\underline{S}}(t))$?

By the chain rule we have

$$\frac{d}{dt}\psi(\underline{\underline{S}}(t)) = \frac{d}{dt}\psi(S_{11}(t), S_{12}(t), \dots, S_{33}(t))$$
$$= \frac{\partial \psi}{\partial S_{11}}\frac{dS_{11}}{dt} + \dots + \frac{\partial \psi}{\partial S_{33}}\frac{dS_{33}}{dt} = \frac{\partial \psi}{\partial S_{ij}}\frac{dS_{ij}}{dt}$$
$$= D\psi(\underline{\underline{S}}) : \dot{\underline{\underline{S}}}$$

$\Rightarrow$ chain rule leads to a contraction

$$\boxed{\frac{d}{dt}\psi(\underline{\underline{S}}(t)) = D\psi(\underline{\underline{S}}) : \dot{\underline{\underline{S}}}}$$

Example: $\boxed{\frac{d}{dt}\det(\underline{\underline{S}}(t)) = \det(\underline{\underline{S}})\, \underline{\underline{S}}^{-T} : \dot{\underline{\underline{S}}}}$ Jacobi's formula