Adiabatic decompression

Internal energy: $du = c_p \, dT - \frac{\alpha T}{\rho} dp$

$\Rightarrow$ change with pressure

material derivative: $\dot{u} = c_p \dot{T} - \frac{\alpha T}{\rho} \dot{p}$

energy balance: $\rho \dot{u} + \nabla \cdot \underline{q} = \rho H$

$$\rho c_p \dot{T} - \alpha T \dot{p} + \nabla \cdot \underline{q} = \rho H$$

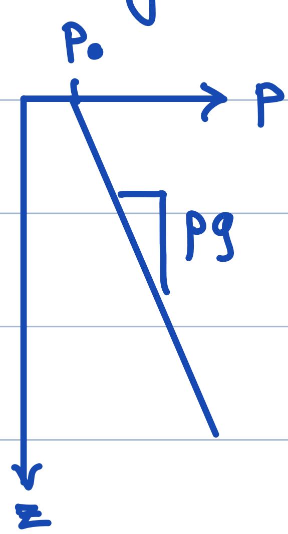
$$\rho c_p \dot{T} + \nabla \cdot \underline{q} = \underbrace{\alpha T \dot{p}}_{\text{new term}} + \rho H$$

In many slow flows, pressure is close to hyrostatic

$p \approx p_0 + \rho g z \qquad z = \text{depth}$

$$\dot{p} = \frac{\partial p}{\partial t} + \underline{v} \cdot \nabla p = 0 + v_z \rho g$$

$\uparrow$ vertical velocity



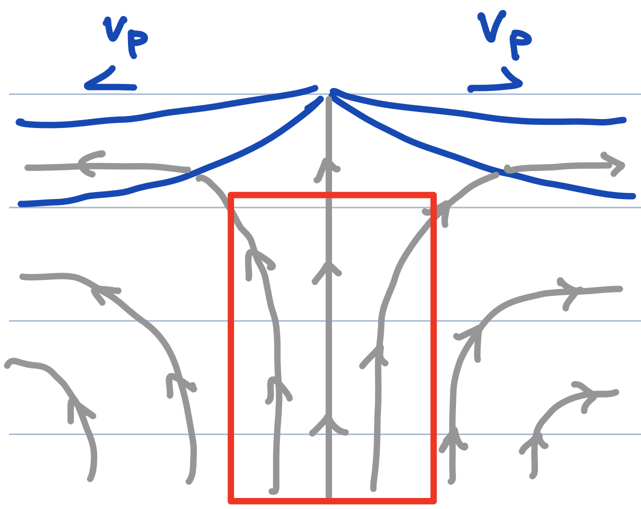
$$\rho c_p \dot{T} + \nabla \cdot \underline{q} = \alpha T v_z \rho g + \rho H$$

$\Rightarrow$ conservative form + Fourier's log

$$\boxed{\rho c_p \frac{\partial T}{\partial t} + \nabla \cdot (\rho c_p T \underline{v} - k \nabla T) = \underbrace{\alpha \rho g v_z T}_{\text{adiabatic heating}} + \rho H}$$

Example: Mantle upwelling beneath MOR

MOR = mid-ocean ridge



beneath ridge mantle flows upward approx vertically with $v_3 = -v_p$

Assume: 1) steady $\frac{\partial T}{\partial t} = 0$

2) conduction is negligible $\nabla \cdot q = 0$

3) radiogenic heating is negligible $\rho H = 0$

4) all properties are constant

5) $\underline{v} = \begin{bmatrix} 0 \\ 0 \\ -v_p \end{bmatrix}$ $\quad v_3 = -v_p$

$$\rho c_p \dot{T} + \cancelto{0}{\nabla \cdot q} = \alpha \rho g v_3 T + \cancelto{0}{\rho H}$$

$$\rho c_p \left(\cancelto{0}{\frac{\partial T}{\partial t}} + \underline{v} \cdot \nabla T \right) = \alpha \rho g v_3 T$$

$$\cancel{\rho} c_p \cancel{v_3} \frac{\partial T}{\partial z} = \alpha \cancel{\rho} g \cancel{v_3} T$$

$$\boxed{\frac{\partial T}{\partial z} = \frac{\alpha g}{c_p} T}$$

adiabatic T gradient

(Note: sign changes with z-axis!)

Earth mantle:

$$\left.\begin{aligned} \alpha &= 3\cdot 10^{-5}\ \frac{1}{K} \\ c_p &= 1200\ \frac{J}{kg\,K} \\ T &= 1500\ K \\ g &= 9.81\ \frac{m}{s^2} \end{aligned}\right\} \quad \frac{\partial T}{\partial z} \approx 4\cdot 10^{-4}\ \frac{K}{m} = 0.4\ \frac{K}{km}$$

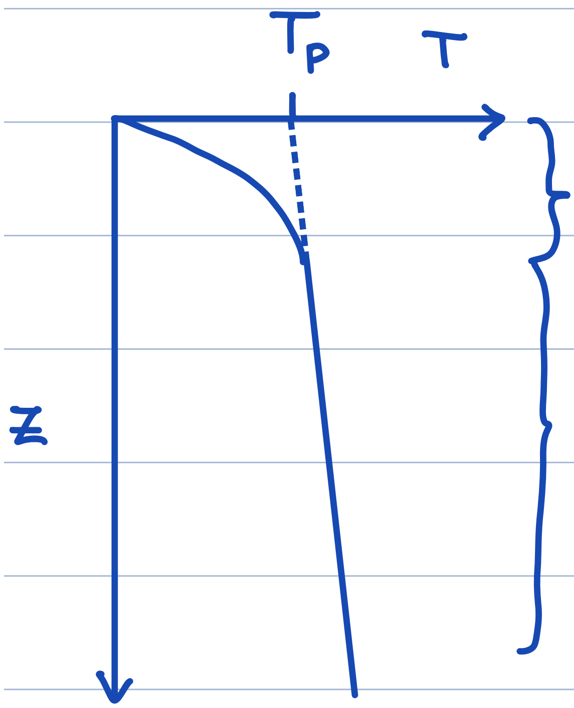
mantle cools $\frac{1}{2}$ degree per kilometer of upwelling

ODE:

$$\frac{dT}{dz} = \frac{\alpha g}{c_p} T$$

$$\frac{dT}{T} = \frac{\alpha g}{c_p} dz \quad \Rightarrow \quad T = c\, e^{\frac{\alpha g}{c_p} z}$$

BC: $T(z=0) = T_p \quad \Rightarrow \quad \boxed{T(z) = T_p\, e^{\frac{\alpha g}{c_p} z} \approx T_p \left(1 - \frac{\alpha g}{c_p} z\right)}$



T_p = mantle potential temperature

Thermal boundary layer

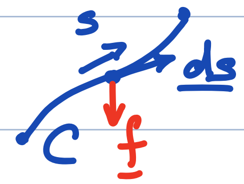
matle adiabat

Work and Power

Work is energy transferred by application of force along distance

$W = F s$

$$W = \int_C \underline{f} \cdot d\underline{s} = \int_{t_1}^{t_2} \underline{f} \cdot \frac{d\underline{s}}{dt}\, dt = \int_{t_1}^{t_2} \underline{f} \cdot \underline{v}\, dt$$



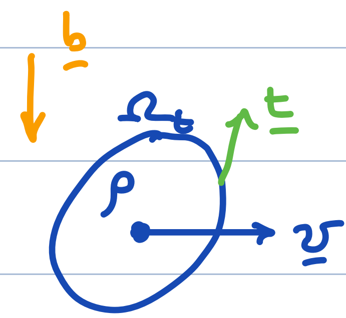
Power is the rate of work

$$\boxed{\mathcal{P} = \frac{dW}{dt} = \underline{f} \cdot \underline{v}}$$

from $W = \int_{t_1}^{t_2} \frac{dW}{dt}\, dt = \int_{t_1}^{t_2} \underline{f} \cdot \underline{v}\, dt$

Working is rate of work done ~ power

Exchange of mechanical energy



Net working $\mathcal{W}[\Omega_t]$ of external forces on Ω_t is the mechanical power not converted into motion

$$\boxed{\mathcal{W}[\Omega_t] = \mathcal{P}[\Omega_t] - \frac{d}{dt}\mathcal{K}[\Omega_t]}$$

Kinetic Energy: $\mathcal{K}[\Omega_t] = \int_\Omega \frac{1}{2} \rho |\underline{v}|^2\, dV$

Power of external forces: $\mathcal{P}[\Omega_t] = \int_{\Omega_t} \rho\, \underline{g} \cdot \underline{v}\, dV + \int_{\partial\Omega_t} \underline{t} \cdot \underline{v}\, dA$

Net working in Eulerian form

Power: $\mathcal{P} = \underline{f} \cdot \underline{v}$

Newton's 2^{nd} law: $\underline{f} = m\underline{a} \rightarrow \underline{f} = \frac{d}{dt}(m\underline{v}) = m\dot{\underline{v}}$

lin. mom. balance: $\rho\dot{\underline{v}} = \nabla \cdot \underline{\underline{\sigma}} + \rho\underline{b}$

Start by taking dot product of $\underline{v}$ and lin. mom. balance

$$\rho\dot{\underline{v}} \cdot \underline{v} = \rho\underline{v} \cdot \dot{\underline{v}} = (\nabla \cdot \underline{\underline{\sigma}}) \cdot \underline{v} + \rho\underline{b} \cdot \underline{v}$$

Identify $\mathcal{K}$, $\mathcal{P}$ and $\mathcal{W}$ => integrate

$$\int_{\Omega_t} \rho\underline{v} \cdot \dot{\underline{v}}\, dV_x = \int_{\Omega_t} (\nabla_x \cdot \underline{\underline{\sigma}}) \cdot \underline{v} + \rho\underline{b} \cdot \underline{v}\, dV_x$$

use identity $\nabla \cdot (\underline{\underline{A}}^T \underline{b}) = (\nabla \cdot \underline{\underline{A}}) \cdot \underline{b} + \underline{\underline{A}} : \nabla\underline{b}$ (Lecture 10)

$$\int_{\Omega_t} \rho\underline{v} \cdot \dot{\underline{v}}\, dV_x = \int_{\Omega_t} -\underline{\underline{\sigma}} : \nabla_x\underline{v} + \nabla \cdot (\underline{\underline{\sigma}}^T \underline{v}) + \rho\underline{b} \cdot \underline{v}\, dV$$

Using $\underline{\underline{S}} : \underline{\underline{D}} = \underline{\underline{S}} : \mathrm{sym}(\underline{\underline{D}})$ if $\underline{\underline{S}} = \underline{\underline{S}}^T$

$\Rightarrow$ introduce $\underline{\underline{d}} = \mathrm{sym}(\nabla_x\underline{v}) = \frac{1}{2}(\nabla_x\underline{v} + \nabla_x\underline{v}^T)$.

$$\int_{\Omega_t} \rho\underline{v} \cdot \dot{\underline{v}}\, dV_x = \int_{\Omega_t} -\underline{\underline{\sigma}} : \underline{\underline{d}} + \rho\underline{b} \cdot \underline{v} + \nabla \cdot (\underline{\underline{\sigma}}^T \underline{v})\, dV$$

use standard div. theorem:

$$\int_{\Omega_t} \rho\underline{v} \cdot \dot{\underline{v}}\, dV_x = \int_{\Omega_t} -\underline{\underline{\sigma}} : \underline{\underline{d}} + \rho\underline{b} \cdot \underline{v}\, dV + \int_{\partial\Omega_t} \underline{\underline{\sigma}}\,\underline{v} \cdot \underline{n}\, dA$$

Using $\underline{\underline{\sigma}}\,\underline{v}\cdot\underline{n} = \underline{v}\cdot\underline{\underline{\sigma}}^T\underline{n} = \underline{v}\cdot\underline{t}$

$$\int_{\Omega_t}\rho\,\underline{v}\cdot\dot{\underline{v}}\,dV_x = \int_{\Omega_t} -\underline{\underline{\sigma}}:\underline{\underline{d}}\,dV + \underbrace{\int_{\Omega_t}\rho\,\underline{b}\cdot\underline{v}\,dV_x + \int_{\partial\Omega_t}\underline{t}\cdot\underline{v}\,dA}_{\mathcal{P}[\Omega_t]}$$

Now we can identify the left hand side as

$$\frac{d}{dt}\mathcal{K}[\Omega_t] = \frac{d}{dt}\int_{\Omega_t}\frac{1}{2}\rho\,\underline{v}\cdot\underline{v}\,dV = \frac{1}{2}\int_{\Omega_t}\rho\frac{d}{dt}(\underline{v}\cdot\underline{v})\,dV$$

$$\frac{d}{dt}(\underline{v}\cdot\underline{v}) = \dot{\underline{v}}\cdot\underline{v} + \underline{v}\cdot\dot{\underline{v}} = 2(\underline{v}\cdot\dot{\underline{v}})$$

$$\Rightarrow \frac{d}{dt}\mathcal{K}[\Omega_t] = \int_{\Omega_t}\rho\,\underline{v}\cdot\dot{\underline{v}}\,dV_x$$

substituting $\mathcal{K}$ and $\mathcal{P}$ we have

$$\boxed{\frac{d}{dt}\mathcal{K}[\Omega_t] = -\int_{\Omega_t}\underline{\underline{\sigma}}:\underline{\underline{d}}\,dV_x + \mathcal{P}[\Omega_t]}$$

by comparison with $\mathcal{W}[\Omega_t] = \mathcal{P}[\Omega_t] - \frac{d}{dt}\mathcal{K}[\Omega_t]$

$$\Rightarrow \boxed{\mathcal{W}[\Omega_t] = \int_{\Omega_t}\underline{\underline{\sigma}}:\underline{\underline{d}}\,dV_x}$$

The quantity $\underline{\underline{\sigma}}:\underline{\underline{d}}$ is called the stress power associated with a motion. It corresponds to the rate of work done by internal forces (stresses) in a continuum body.

Energy balance with net working

$$\frac{d}{dt}\mathcal{U}[\Omega_t] = Q[\Omega_t] + \mathcal{W}[\Omega_t]$$

where $\mathcal{U}[\Omega_t] = \int_{\Omega_t} \rho u \, dV_x$

$$Q[\Omega_t] = \int_{\Omega_t} \rho r \, dV_x - \int_{\partial\Omega_t} \underline{q}\cdot\underline{n}\, dA_x$$

$$\mathcal{W}[\Omega_t] = \int_{\Omega_t} \underline{\underline{\sigma}} : \underline{\underline{d}} \, dV_x$$

Hence we have

$$\frac{d}{dt}\int_{\Omega_t} \rho u \, dV_x = \int_{\Omega_t} \underline{\underline{\sigma}} : \underline{\underline{d}} \, dV_x - \int_{\partial\Omega_t} \underline{q}\cdot\underline{n}\, dA_x + \int_{\Omega_t} \rho r \, dV_x$$

using derivative relative to mass and divergence Thm

$$\int_{\Omega} (\rho \dot{u} - \underline{\underline{\sigma}} : \underline{\underline{d}} + \nabla_x \cdot \underline{q} + \rho r) \, dV_x$$

by the arbitraryness of Ω_t we have

$$\boxed{\rho \dot{u} = \underbrace{\underline{\underline{\sigma}} : \underline{\underline{d}}}_{\text{networking}} - \nabla_x \cdot \underline{q} + \rho r}$$ local Eulerian form

$$\boxed{\frac{\partial}{\partial t}(\rho u) + \nabla_x \cdot [\underline{v} \rho u + \underline{q}] = \underline{\underline{\sigma}} : \underline{\underline{d}} + \rho r}$$ conservative local Eulerian form