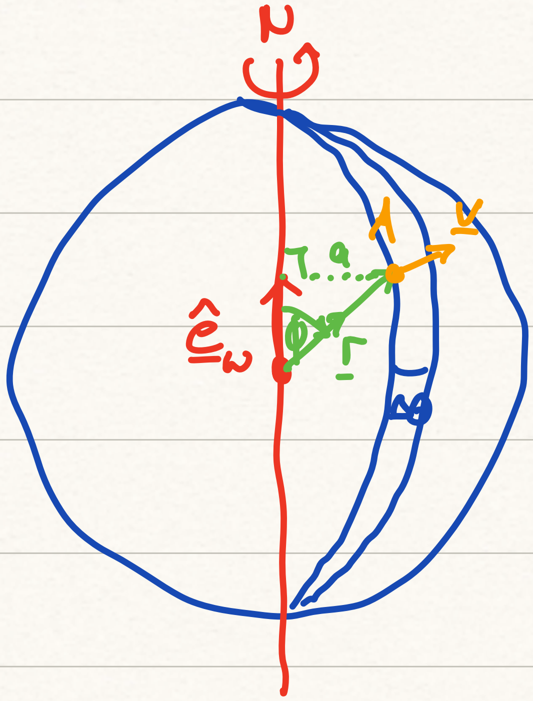
Angular momentum & torque

Rotational motion:



Angular velocity: $\underline{\omega} = |\underline{\omega}|\, \hat{\underline{e}}_\omega$

$$|\underline{\omega}| = \frac{d\theta}{dt}$$

Position vector: $\underline{r} = |\underline{r}|\, \hat{\underline{e}}_r$

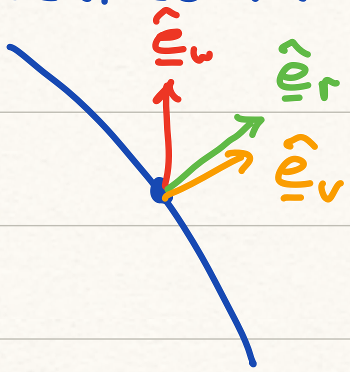
Velocity: $\underline{v} = |\underline{v}|\, \hat{\underline{e}}_v$

$$|\underline{v}| = |\underline{\omega}|\, a = |\underline{\omega}||\underline{r}| \sin\phi$$

$$a = |\underline{r}| \sin\phi$$

$$\Rightarrow \underline{v} = |\underline{\omega}|\,|\underline{r}| \sin\phi \;\hat{\underline{e}}_v$$

local coord: $\hat{\underline{e}}_v \perp \hat{\underline{e}}_\omega$ & $\hat{\underline{e}}_v \perp \hat{\underline{e}}_r$



$$\Rightarrow \hat{\underline{e}}_\omega \times \hat{\underline{e}}_r = \sin\phi\, \hat{\underline{e}}_v$$

substitute

$$\underline{v} = |\underline{\omega}||\underline{r}|\, \hat{\underline{e}}_\omega \times \hat{\underline{e}}_r$$

$$= |\underline{\omega}|\hat{\underline{e}}_\omega \times |\underline{r}|\hat{\underline{e}}_r = \underline{\omega} \times \underline{r}$$

counterclockwise

$|v| > 0$

$$\boxed{\underline{v} = \underline{\omega} \times \underline{r}}$$

"right hand rule"

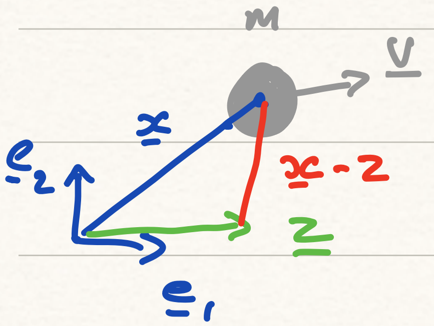
Example: $\phi \approx 60° = \frac{\pi}{3} \approx 1.05$ $\qquad |\underline{r}| = 6.37 \cdot 10^6\, m$

$|\omega| = 7.3 \cdot 10^{-5}\, \frac{rad}{s}$ $\qquad |\underline{v}| = |\underline{\omega}||\underline{r}| \sin\phi \approx 403 \frac{m}{s}$

Angular momentum & torque

Linear momomentum: $\boxed{\underline{L} = m\underline{v}}$

angular momentum: $\boxed{\underline{j} = (\underline{x} - \underline{z}) \times \underline{L}}$



relative to $\underline{z}$

force ⇒ change in lin. mom.

torque ⇒ change in ang. mom.

torque: $$\underline{\tau} = \frac{d\underline{j}}{dt} = \frac{d}{dt}\left[(\underline{x} - \underline{z}) \times m\,\underline{v}\right]$$

$$= m\,\frac{d}{dt}\left[\underline{x} \times \underline{v} - \underline{z} \times \underline{v}\right]$$

$\frac{d\underline{x}}{dt} = \underline{v}$ $\quad \frac{d\underline{v}}{dt} = \underline{a}$

$$= m\left[\cancelto{0}{\underline{v} \times \underline{v}} + \underline{x} \times \underline{a} - \underline{z} \times \underline{a}\right]$$

$$= m\,(\underline{x} - \underline{z}) \times \underline{a} = (\underline{x} - \underline{z}) \times \underbrace{m\,\underline{a}}_{f}$$

$$\boxed{\underline{\tau} = (\underline{x} - \underline{z}) \times \underline{f}}$$

units: $\left[F\,L = \frac{ML^2}{T^2}\right]$ $\quad$ Nm

torque = moment of force, moment

Two basic relations:

1) ang. & lin. mom $\qquad \underline{j} = (\underline{x} - \underline{z}) \times \underline{L}$

2) torque & force $\qquad \underline{\tau} = (\underline{x} - \underline{z}) \times \underline{f}$

Resultant torque due to

1) body forces: $\qquad \underline{\tau}_b = \int_B (\underline{x} - \underline{z}) \times \underline{b} \, dV$

2) surface forces: $\qquad \underline{\tau}_s = \int_\Gamma (\underline{x} - \underline{z}) \times \underline{t}_n \, dA$

In rotation we have important geom. locations:

center of volume: $\qquad \underline{x}_V = \frac{1}{V_B} \int_B \underline{x} \, dV$

center of mass: $\qquad \underline{x}_m = \frac{1}{m_b} \int_B \rho \underline{x} \, dV$

$\rho = \text{const}$: $\qquad \underline{x}_m = \underline{x}_V$

Torque due to grav. body force

torque around $\underline{x}_m = \text{const.}$ $\quad$ $\underline{g} = \text{const.}$

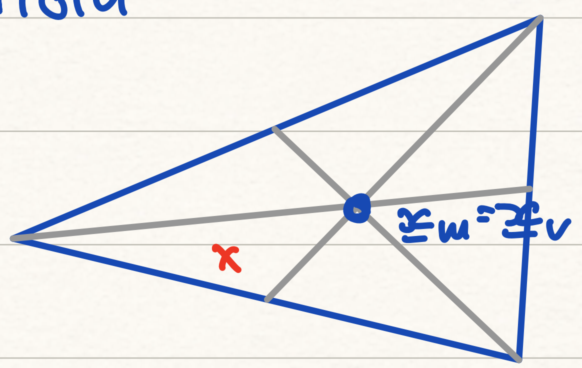
$$\underline{\tau}_b = \int_B (\underline{x} - \underline{x}_m) \times \rho \underline{g} \, dV$$
$$= \int_B \underline{x} \times \rho \underline{g} \, dV - \int_B \underline{x}_m \times \rho \underline{g} \, dV$$

$$= \underbrace{\int_B \underline{x} \rho \, dV}_{m_b \underline{x}_m} \times \underline{g} - \underline{x}_m \times \underline{g} \underbrace{\int_B \rho \, dV}_{m_b}$$

$$\underline{\tau}_b = m_b \underline{x}_m \times \underline{g} - \underline{x}_m \times m_b \underline{g}$$
$$= m_b (\underline{x}_m \times \underline{g} - \underline{x}_m \times \underline{g}) = \underline{0}$$

$\Rightarrow$ grav. torque around center of mass is zero

centroid



B Triangle suspended from centroid does not rotate

Moment of gravity

torque due gravity around origin $\underline{x} = \underline{0}$

$$\boxed{\underline{\tau}_G = \int_B \underline{x} \times \rho_b \underline{g} \, dV}$$

simplify

$$\underline{\tau}_G = \int_B (\underline{x} - \underline{x}_m + \underline{x}_m) \times \rho_b \underline{g} \, dV$$

$$= \cancelto{0}{\int_B (\underline{x} - \underline{x}_m) \times \rho_b \underline{g} \, dV} + \int_B \underline{x}_m \times \rho \underline{g} \, dV$$

$$= \int_B \underline{x}_m \times \rho_b \underline{g} \, dV = \underline{x}_m \times \underline{g} \underbrace{\int_B \rho_b \, dV}_{m_b}$$

$$\boxed{\underline{\tau}_G = \underline{x}_m \times m_b \underline{g}}$$ Moment of Gravity

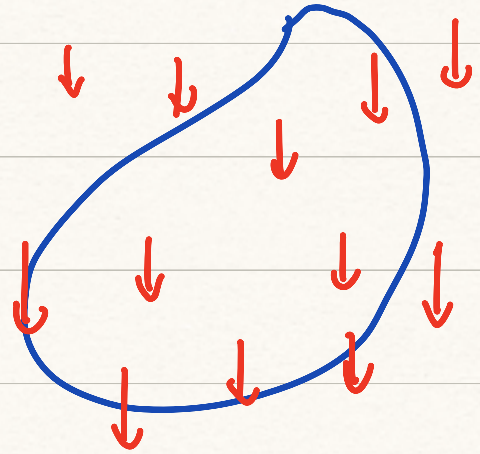
center of mass $\Longleftrightarrow$ center of gravity

$$\underline{x}_m = \underline{x}_G$$

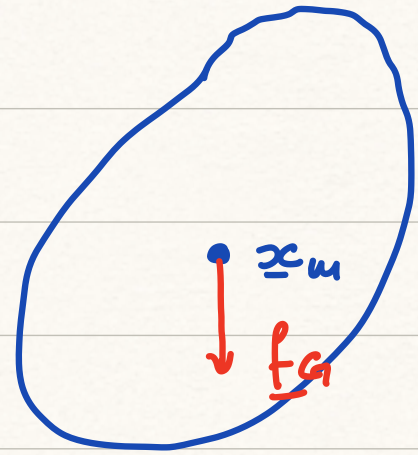
$\Rightarrow$ Gravity acts on center of mass

Continuum

$\underline{b}_G(\underline{x}) = \rho_b(\underline{x})\,\underline{g}$



Discrete



force field can be represented as acting on the point where it does not induce a torque! (Center of mass theorem)

Hydrostatic surface force

"Moment of buoyancy"

torque due to buoyancy around origin ($\underline{x}=\underline{0}$)

$$\boxed{\underline{T}_B = \oint_{\partial B} \underline{x} \times (-p\,\underline{\hat{n}})\,dA}$$

$\underline{t}_n = -p\,\underline{\hat{n}}$

torque due to buoyancy vanishes at center of mass of displaced fluid

$$\underline{x}_B = \frac{1}{V_B}\int_B \rho_f(\underline{x})\,\underline{x}\,dV \equiv \underline{0}$$

use to simplify moment of buoyancy

$$\boxed{\underline{\tau}_B = -\underline{x}_B \times m_f\,\underline{g}} \qquad \Rightarrow \text{HW2}$$

Hydrostatic moment

force balance: $\underline{f}_H = \underline{f}_G + \underline{f}_B = (m_b - m_f)\,\underline{g}$

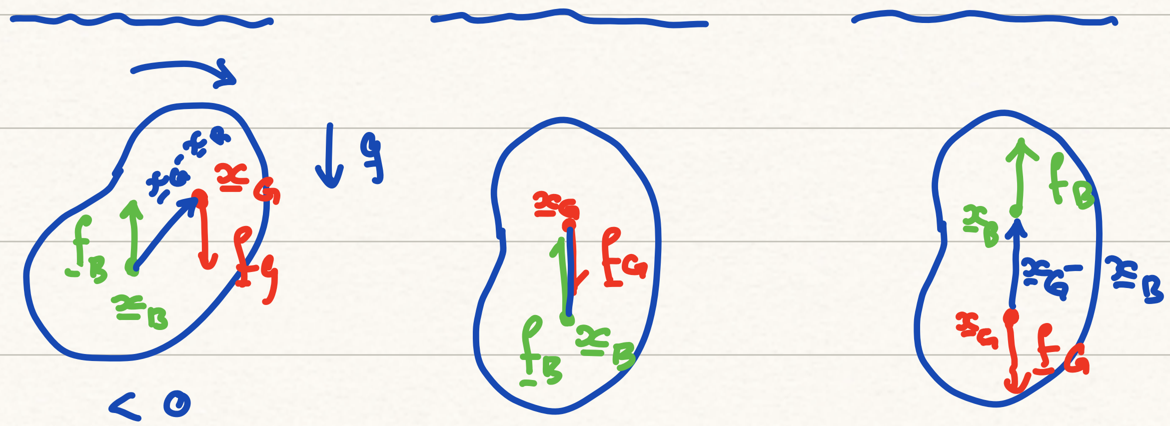
torque balance: $\underline{\tau}_H = \underline{\tau}_G + \underline{\tau}_B =$

$$= \underline{x}_G \times m_b\,\underline{g} - \underline{x}_B \times m_f\,\underline{g}$$

neutrally buoyant body: $m_f = m_b = m$

$$\boxed{\underline{\tau}_H = m\,\underbrace{(\underline{x}_G - \underline{x}_B)}_{\neq 0} \times \underbrace{\underline{g}}_{\neq 0}}$$

Stability of fully submerged floating body



unstable	metastable	stable
$\underline{\tau} \neq 0$ (< 0)	$\underline{\tau} = \underline{0}$	$\underline{\tau} = 0$

unstable → roll clockwise