Lecture 4: Tensors

Logistics: - HW1 due

- HW2 due next Thursday

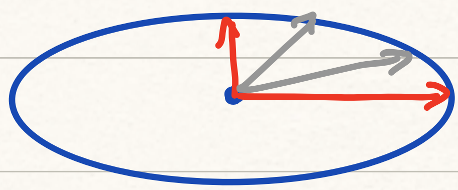
Scalar: quantity at a point

→ Temperature,

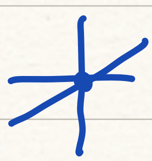
Vectors: quantity & direction at ~~any~~ a point

→ velocity, force

Tensors: how a quantity changes with direction at a point



anisotropic physial properties



stress, strain, moment of inertia

Second-order tensors

→ matrix

Linear operators: $\underline{v} = \underline{\underline{A}}\,\underline{u}$

$\uparrow$ tensor

Special tensors:

Zero tensor: $\underline{\underline{0}}\,\underline{v} = \underline{0}$ for all $\underline{v}$

Identity tensor: $\underline{\underline{I}}\,\underline{v} = \underline{v}$ " " "

Basic algebra:

1) scalar multiplication: $(\alpha \underline{\underline{A}})\underline{v} = \underline{\underline{A}}(\alpha \underline{v})$

2) tensor sum: $(\underline{\underline{A}} + \underline{\underline{B}})\underline{v} = \underline{\underline{A}}\,\underline{v} + \underline{\underline{B}}\,\underline{v}$

3) Tensor product: $(\underline{\underline{A}}\,\underline{\underline{B}})\underline{v} = \underline{\underline{A}}(\underline{\underline{B}}\,\underline{v})$

(4) Tensor scalar product)

$\left.\begin{array}{l} 1+2 \Rightarrow \text{imply linearity} \\ 1+2+3 \quad \text{produce tensors} \end{array}\right\}$ set $\mathcal{V}^2$ of tensors is a vector space

Q: Basis for $\mathcal{V}^2$?

Representation of a tensor

frame $\{\underline{e}_i\}$

$$\boxed{S_{ij} = \underline{e}_i \cdot \underline{\underline{S}}\,\underline{e}_j}$$

Matrix representation

$$[\underline{\underline{S}}] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

$\Rightarrow$ close analogy to matrix manipulation

Example: $\underline{v} = \underline{\underline{S}}\,\underline{u}$ $\qquad \underline{v} = v_k \underline{e}_k \quad \underline{u} = u_j \underline{e}_j$

$$v_k \underline{e}_k = \underline{\underline{S}}\, u_j \underline{e}_j$$

$$v_k \underbrace{\underline{e}_i \cdot \underline{e}_k} = \underbrace{\underline{e}_i \cdot \underline{\underline{S}}\,\underline{e}_j}\, u_j$$

$$v_k\, \delta_{ik} = S_{ij}\, u_j$$

$$\boxed{v_i = S_{ij} u_j} \longleftrightarrow \underline{v} = \underline{\underline{S}}\,\underline{u}$$

$$\underbrace{v_i \underline{e}_i}_{\underline{v}} = \underbrace{S_{ij} u_j \underline{e}_i}_{\underline{\underline{S}}\,\underline{u}}$$

Dyadic Product

$$\boxed{(\underline{a} \otimes \underline{b})\,\underline{v} = (\underline{b} \cdot \underline{v})\,\underline{a}}$$

$$\underline{\underline{S}}\ \underline{v} = \alpha\ \underline{a} \qquad \alpha = b_j v_j$$

vector

$\Rightarrow$ $\underline{a} \otimes \underline{b}$ is tensor !

$$S_{ij} v_j = (b_j v_j)\, a_i$$

$$\underbrace{[\underline{a} \otimes \underline{b}]}_{}{}_{ij}\, v_j = b_j v_j\, a_i$$

$$= \underbrace{a_i\, b_j}_{}\, v_j$$

$$\Rightarrow \boxed{[\underline{a} \otimes \underline{b}]_{ij} = a_i b_j}$$

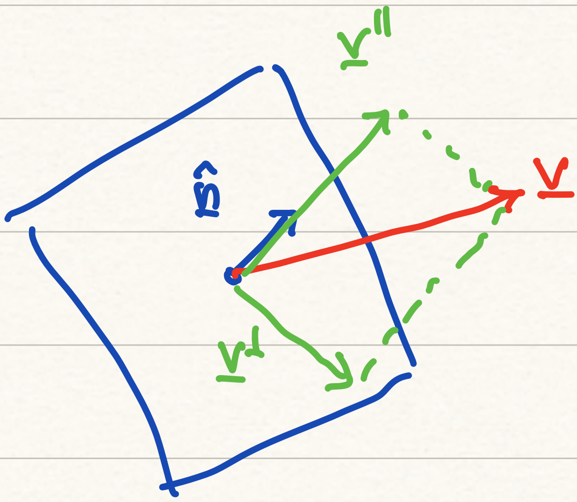
$$[\underline{a} \otimes \underline{b}] = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \left(= \underline{a}\, \underline{b}^T \right)$$

Linearity:

$$\boxed{(\underline{a} \otimes \underline{b})(\alpha \underline{v} + \beta \underline{w}) = \alpha (\underline{a} \otimes \underline{b})\,\underline{v} + \beta (\underline{a} \otimes \underline{b})\,\underline{w}}$$

$$\alpha(\underline{a} \otimes \underline{b}) = (\alpha \underline{a} \otimes \underline{b}) = (\underline{a} \otimes \alpha \underline{b})$$

Projection tensor



$\underline{v} = \underline{v}^{\parallel} + \underline{v}^{\perp}$

use dot product

$\underline{v}^{\parallel} = \underbrace{(\underline{v} \cdot \hat{\underline{n}})\, \hat{\underline{n}}}$

$\underline{v}^{\perp} = \underline{v} - \underline{v}^{\parallel} \quad = -(\underline{v} \times \hat{\underline{n}} \times \hat{\underline{n}})$

$$\underline{v}^{\parallel} = \underline{\underline{P}}^{\parallel}\, \underline{v}$$

Tensors?

$$\underline{v}^{\perp} = \underline{\underline{P}}^{\perp}\, \underline{v}$$

What are these tensors?

use dyadic property $(\underline{a} \otimes \underline{b})\, \underline{v} = \underbrace{(\underline{b} \cdot \underline{v})\, \underline{a}}$

$\underline{b} = \hat{\underline{n}} \qquad \underline{a} = \hat{\underline{n}}$

$\Rightarrow \quad (\hat{\underline{n}} \cdot \underline{v})\, \hat{\underline{n}} = (\hat{\underline{n}} \otimes \hat{\underline{n}})\, \underline{v} = \underline{v}^{\parallel}$

$\Rightarrow \quad \underline{\underline{P}}^{\parallel} = \hat{\underline{n}} \otimes \hat{\underline{n}}$

$\underline{v}^{\perp} = \underline{v} - \underline{v}^{\parallel} = \underline{\underline{I}}\, \underline{v} - \underline{\underline{P}}^{\parallel} \underline{v} = (\underline{\underline{I}} - \underline{\underline{P}}^{\parallel})\, \underline{v}$

$\underline{\underline{P}}^{\perp} = \underline{\underline{I}} - \underline{\underline{P}}^{\parallel} = \underline{\underline{I}} - \hat{\underline{n}} \otimes \hat{\underline{n}}$

$$\boxed{\underline{\underline{P}}^{\parallel} = \hat{\underline{n}} \otimes \hat{\underline{n}} \qquad \underline{\underline{P}}^{\perp} = \underline{\underline{I}} - \hat{\underline{n}} \otimes \hat{\underline{n}}}$$

Basis of $\mathcal{V}^2$

frame $\{\underline{e}_i\}$ → nine dyadic products $\{\underline{e}_i \otimes \underline{e}_j\}$
→ basis for $\mathcal{V}^2$

$$\boxed{\underline{\underline{S}} = S_{ij}\, \underline{e}_i \otimes \underline{e}_j} \qquad S_{ij} = \underline{e}_i \cdot \underline{\underline{S}}\, \underline{e}_j$$

comp ↑ ↑ basis

analogous: $\underline{v} = v_i\, \underline{e}_i \qquad v_i = \underline{v} \cdot \underline{e}_i$

comp ↑ ↑ Basis

Example: $\underline{v} = \underline{\underline{S}}\, \underline{u} \qquad \underline{v} = v_i\, \underline{e}_i \qquad \underline{u} = u_k\, \underline{e}_k$

$$\begin{aligned} v_i\, \underline{e}_i &= S_{ij}\, \underline{e}_i \otimes \underline{e}_j\, u_k\, \underline{e}_k \\ &= S_{ij}\, u_k\, (\underline{e}_i \otimes \underline{e}_j)\, \underline{e}_k \qquad \text{dyadic prod.} \\ &= S_{ij}\, u_k\, \underbrace{(\underline{e}_j \cdot \underline{e}_k)}_{\delta_{jk}}\, \underline{e}_i \end{aligned}$$

$$v_i\, \underline{e}_i = \underline{S_{ij}\, u_j}\, \underline{e}_i \qquad v_i = S_{ij}\, u_j$$

Product of dyadic products

$$\boxed{(\underline{a} \otimes \underline{b})(\underline{c} \otimes \underline{d}) = (\underline{b} \cdot \underline{c})(\underline{a} \otimes \underline{d})} \quad \Rightarrow \text{HW3}$$

⇒ tensor product

Example: $\underline{\underline{H}} = \underline{\underline{S}}\,\underline{\underline{T}}$

$$\underline{\underline{H}} = S_{ij}(\underline{e}_i \otimes \underline{e}_j)\, T_{kl}\,(\underline{e}_k \otimes \underline{e}_l)$$

$$= S_{ij} \cdot T_{kl}\,(\underline{e}_i \otimes \underline{e}_j)(\underline{e}_k \otimes \underline{e}_l) \qquad \text{dyadic p.}$$

$$= S_{ij}\, T_{kl}\, \underbrace{(\underline{e}_j \cdot \underline{e}_k)}_{\delta_{jk}}(\underline{e}_i \otimes \underline{e}_l)$$

$$= S_{ij}\, T_{jl}\; \underline{e}_i \otimes \underline{e}_l$$

$$\underline{\underline{H}} = H_{il}(\underline{e}_i \otimes \underline{e}_l) = S_{ij}\, T_{jl}\,(\underline{e}_i \otimes \underline{e}_l)$$

$$\Rightarrow \boxed{H_{il} = S_{ij}\, T_{jl}}$$

recognize tensor product by dummy in middle

Determinant

$$\boxed{\det(\underline{A}) = \det\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = \epsilon_{ijk}\, [\underline{A}]_{i1}\, [\underline{A}]_{j2}\, [\underline{A}]_{k3}}$$

where $[A]_{i1}$ is first column of $[\underline{A}]$

$\Rightarrow$ triple scalar product $(\underline{a} \times \underline{b}) \cdot \underline{c} = \det([\underline{a}, \underline{b}, \underline{c}])$

$$= \epsilon_{ijk}\, a_i\, b_j\, c_k$$

$\Rightarrow$ volumes

strain tensor : $\underline{\underline{E}}$ $\quad$ $\det(\underline{\underline{E}})$ volume change

Properties: $\det(\underline{\underline{A}}\,\underline{\underline{B}}) = \det(\underline{\underline{A}})\det(\underline{\underline{B}})$

$\det(\underline{\underline{A}}^T) = \det(\underline{\underline{A}})$

$\det(\alpha \underline{\underline{A}}) = \alpha^n \det(A)$ $\quad$ n = dimension
3

$\underline{\underline{A}}$ is singular if $\det(\underline{\underline{A}}) = 0$

$\det(\underline{\underline{A}}) \neq 0$:

$$\boxed{\underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \underline{\underline{I}}}$$

Transpose

$$\boxed{\underline{\underline{S}}\,\underline{u} \cdot \underline{v} = \underline{u} \cdot \underline{\underline{S}}^T \underline{v}}$$

$$(S_{ij} u_j \underline{e}_i) \cdot (v_l \underline{e}_l) = (u_k \underline{e}_k) \cdot (S^T_{ij} v_j \underline{e}_i)$$

$$S_{ij} u_j v_l \underbrace{(\underline{e}_i \cdot \underline{e}_l)}_{\delta_{il}} = S^T_{ij} v_j u_k \underbrace{(\underline{e}_k \cdot \underline{e}_i)}_{\delta_{ki}}$$

$$S_{ij} u_j v_i = S^T_{ij} v_j u_i$$

$$S_{ij}\, u_j v_i = S^T_{ij}\, v_j u_i$$

lhs: $j \leftrightarrow i$

$$S_{ji}\, u_i v_j = S^T_{ij}\, u_i v_j$$

$\Longrightarrow$ $\boxed{S^T_{ij} = S_{ji}}$ transpose in components

$$\underline{\underline{S}}^T = \begin{bmatrix} S_{11} & S_{21} & S_{31} \\ S_{12} & S_{22} & S_{32} \\ S_{13} & S_{23} & S_{33} \end{bmatrix} \qquad \underline{\underline{S}} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

Properties of transpose:

$$(\underline{\underline{A}}^T)^T = \underline{\underline{A}}$$

$$(\underline{\underline{A}}\,\underline{\underline{B}})^T = \underline{\underline{B}}^T \underline{\underline{A}}^T$$

$$(\underline{u} \otimes \underline{v})^T = \underline{v} \otimes \underline{u}$$

$\underline{\underline{S}}$ is symmetric if $\underline{\underline{S}} = \underline{\underline{S}}^T$ $\quad S_{ij} = S_{ji}$

$\underline{\underline{S}}$ is skew symmetric if $\underline{\underline{S}} = -\underline{\underline{S}}^T$ $\quad S_{ij} = -S_{ji}$

Symmetric-Skew decomposition

$$\boxed{\begin{aligned} \underline{\underline{S}} &= \underline{\underline{E}} + \underline{\underline{W}} \\ \underline{\underline{E}} &= \tfrac{1}{2}\left(\underline{\underline{S}} + \underline{\underline{S}}^T\right) \\ \underline{\underline{W}} &= \tfrac{1}{2}\left(\underline{\underline{S}} - \underline{\underline{S}}^T\right) \end{aligned}}$$

$$\Rightarrow \quad \underline{\underline{E}} = \underline{\underline{E}}^T$$

$$\Rightarrow \quad \underline{\underline{W}} = -\underline{\underline{W}}^T$$

→ skew tensors are related to rotation & cross products

Trace

$$\boxed{\mathrm{tr}(\underline{a} \otimes \underline{b}) = \underline{a} \cdot \underline{b} = a_i b_i}$$

implies $\quad \mathrm{tr}\,\underline{\underline{A}} = A_{11} + A_{22} + A_{33}$

$$\mathrm{tr}(\underline{\underline{A}}) = \mathrm{tr}(A_{ij}\,\underline{e}_i \otimes \underline{e}_j) = A_{ij}\,\mathrm{tr}(\underline{e}_i \otimes \underline{e}_j) = A_{ij}\,\underbrace{\underline{e}_i \cdot \underline{e}_j}_{\delta_{ij}}$$

$$= A_{ii}$$

Properties: $\quad \mathrm{tr}(\underline{\underline{A}}^T) = \mathrm{tr}(\underline{\underline{A}})$

$$\mathrm{tr}(\underline{\underline{A}}\,\underline{\underline{B}}) = \mathrm{tr}(\underline{\underline{B}}\,\underline{\underline{A}})$$

$$\mathrm{tr}(\underline{\underline{A}} + \underline{\underline{B}}) = \mathrm{tr}(\underline{\underline{A}}) + \mathrm{tr}(\underline{\underline{B}})$$

$$\mathrm{tr}(\alpha \underline{\underline{A}}) = \alpha\, \mathrm{tr}(\underline{\underline{A}})$$

Spherical-Deviatoric Decomposition

$$\underline{\underline{A}} = \alpha \underline{\underline{I}} + \mathrm{dev}(\underline{\underline{A}})$$

Spherical tensor: αI where $\alpha = \frac{1}{3}\mathrm{tr}(A)$

Deviatoric tensor: $\mathrm{dev}(\underline{\underline{A}}) = \underline{\underline{A}} - \alpha \underline{\underline{I}}$

$$\Rightarrow \mathrm{tr}(\mathrm{dev}(A)) = 0$$