Lecture 8: Change in Basis

Logistics: HW 3

$\underline{v}$ & $\underline{\underline{S}}$ unique

$[\underline{v}]$ & $[\underline{\underline{S}}]$ not unique

Two frames $\{\underline{e}_i\}$ and $\{\underline{e}'_i\}$

representation of $\underline{e}'_j$ in $\{\underline{e}_i\}$

$$[\underline{a}] = a_i \underline{e}_i = (\underline{a} \cdot \underline{e}_i)\, \underline{e}_i \qquad a_i = \underline{a} \cdot \underline{e}_i$$

$$[\underline{e}'_j] = (\underline{e}'_j \cdot \underline{e}_1)\underline{e}_1 + (\underline{e}'_j \cdot \underline{e}_2)\underline{e}_2 + (\underline{e}'_j \cdot \underline{e}_3)\underline{e}_3$$

$$= \underbrace{(\underline{e}'_j \cdot \underline{e}_i)}_{A_{ij}}\underline{e}_i$$

$$\boxed{\underline{\underline{A}} = A_{ij}\, \underline{e}_i \otimes \underline{e}_j \qquad A_{ij} = \underline{e}_i \cdot \underline{e}'_j}$$

Change of basis tensor

$\underline{\underline{A}}\,\underline{\underline{B}} = A_{ij} B_{jk}$

what kind of tensor? $\underline{\underline{A}}^T \underline{\underline{B}} \qquad A_{\underline{j}i} B_{\underline{j}k}$

$$[\underline{e}'_j] = (\underline{e}'_j \cdot \underline{e}_i)\,\underline{e}_i = A_{ij}\,\underline{e}_i$$

Express $\underline{e}_i$ in $\{\underline{e}'_k\}$

$$[\underline{e}_i]' = (\underline{e}_i \cdot \underline{e}'_k)\,\underline{e}'_k = A_{ik}\,\underline{e}'_k$$

$$\underline{e}'_j = A_{ij}\,\underline{e}_i$$

$$\underline{e}'_j = A_{ij}\,(A_{ik}\,\underline{e}'_k)$$

$$\underline{e}'_j = \underbrace{A_{ij}\,A_{ik}}_{\delta_{jk}}\,\underline{e}'_k$$

$$\underline{e}'_j = \delta_{jk}\,\underline{e}'_k$$

$$A_{ij}\,A_{ik} = \delta_{jk}$$

$$\underline{\underline{A}}^T\,\underline{\underline{A}} = \underline{\underline{I}}$$

$$\underline{e}_i = A_{ik}\,\underline{e}'_k$$

$$\underline{e}_i = A_{ik}\,(A_{lk}\,\underline{e}_l)$$

$$= \underbrace{A_{ik}\,A_{lk}}_{\delta_{il}}\,\underline{e}_l$$

$$\underline{e}_i = \delta_{il}\,\underline{e}_l$$

$$A_{ik}\,A_{lk} = \delta_{il}$$

$$\underline{\underline{A}}\,\underline{\underline{A}}^T = \underline{\underline{I}}$$

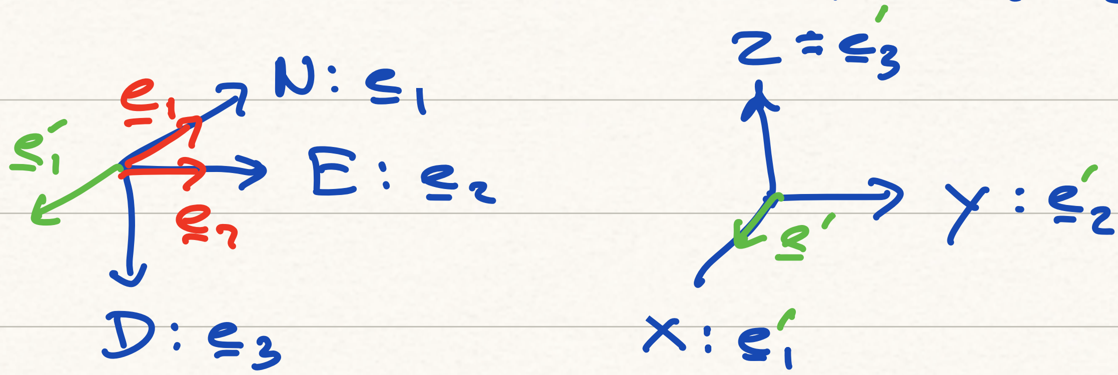
$$\boxed{\underline{\underline{A}}^T\,\underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{A}}^T = \underline{\underline{I}}}$$ orthonormal

$$\underline{\underline{A}}^T = \underline{\underline{A}}^{-1}$$

if $\{\underline{e}_i\}$ and $\{\underline{e}_i'\}$ are right handed

$\Rightarrow$ $\underline{\underline{A}}$ must be a rotation $\quad \det(\underline{\underline{A}}) = 1$

Example: NED $\{\underline{e}_i\}$ and XYZ $\{\underline{e}_i'\}$



$$[\underline{e}_1'] = \underbrace{(\underline{e}_1 \cdot \underline{e}_1')}_{-1}\underline{e}_1 + \underbrace{(\underline{e}_2 \cdot \underline{e}_1')}_{0}\underline{e}_2 + \underbrace{(\underline{e}_3 \cdot \underline{e}_1')}_{0}\underline{e}_3$$

$$[\underline{e}_1'] = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \qquad [\underline{e}_2'] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad [\underline{e}_3'] = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\underline{\underline{A}} = A_{ij}\,\underline{e}_i \otimes \underline{e}_j \qquad A_{ij} = \underline{e}_i \cdot \underline{e}_j'$$

$$\underline{\underline{A}} = \begin{bmatrix} | & | & | \\ [\underline{e}_1'] & [\underline{e}_2'] & [\underline{e}_3'] \\ | & | & | \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Change representation of vector

$$\boxed{[\underline{v}] = [\underline{\underline{A}}][\underline{v}]' \quad \text{and} \quad [\underline{v}]' = [\underline{\underline{A}}]^T[\underline{v}]}$$

$$\underline{v} = v_i \underline{e}_i = v_j' \underline{e}_j' \qquad \text{but} \qquad \underline{e}_j' = A_{ij} \underline{e}_i$$

$$= v_j'(A_{ij} \underline{e}_i)$$

$$= A_{ij} v_j' \underline{e}_i$$

$$\Rightarrow \qquad v_i = A_{ij} v_j'$$

Example: HW1 NED

$$[\underline{a}] = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix} \text{ in NED}$$

$$[\underline{a}]' = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \text{ in XYZ}$$

$$\underline{\underline{A}} = \begin{bmatrix} -1 & & \\ & 1 & \\ & & -1 \end{bmatrix}$$

$$[\underline{\underline{A}}] \cdot [\underline{a}]' = \begin{bmatrix} -1 & & \\ & 1 & \\ & & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix} = [\underline{a}]$$

Change Tensor representation

$$\boxed{\begin{aligned} [\underline{\underline{S}}] &= [\underline{\underline{A}}][\underline{\underline{S}}]'[\underline{\underline{A}}]^T \\ [\underline{\underline{S}}]' &= [\underline{\underline{A}}]^T[\underline{\underline{S}}][\underline{\underline{A}}] \end{aligned}}$$

Look familiar ? → diagonalizing matrix

$$[\underline{\underline{S}} - \lambda \underline{\underline{I}}] \, \underline{v} = 0$$

not diagonal

$\lambda_p, \underline{v}_p$

$\underline{V} = [\underline{v}_1 \ \underline{v}_2 \ \underline{v}_3]$

$$\underline{\underline{V}}^{-1} \underline{\underline{S}} \, \underline{\underline{V}} = \underline{\underline{\Lambda}} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$$

⇒ $\underline{V}$ = change in basis

principal frame ⟷ XYZ

Eigenvalues & Eigenvectors of Tensors

$$\boxed{\underline{\underline{S}}\,\underline{v} = \lambda\,\underline{v}}$$

eigen vec. eigen val.

λ_p roots of char. polynomial

$$\det(\underline{\underline{S}} - \lambda \underline{\underline{I}}) = 0$$

For each λ_p one or more $\underline{v}_p$ that satisfy

$$(\underline{\underline{S}} - \lambda_p \underline{\underline{I}})\,\underline{v}_p = \underline{0}$$

In continuum mech: symmetric tensors

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$$

Eigenproblem for sym. tensors

1) All λ_p's are real
2) All λ_p's are positive $\Rightarrow$ (sym. pos. def.) SPD
3) All $\underline{v}_p$'s corresponding to distinct λ_p's are orthogonal

Sym. pos def.:

$$\underline{v} \cdot \underline{\underline{S}}\,\underline{v} > 0 \qquad \text{for all } \underline{v}$$

$$\underline{v} \cdot \lambda \underline{v} > 0$$

$$\lambda\ \underline{v} \cdot \underline{v} > 0$$

$$\lambda\ \underbrace{|\underline{v}|^2}_{>0} > 0 \quad \rightarrow \quad \lambda > 0$$

Orthogonality of two eigenvectors

$(\lambda, \underline{v})$ $\quad$ $(\omega, \underline{u})$ $\quad$ $\lambda \neq \omega$

$\underline{\underline{S}}\,\underline{v} = \lambda \underline{v}$ $\quad$ $\underline{\underline{S}}\,\underline{u} = \omega \underline{u}$

Consider $\quad \lambda(\underline{v} \cdot \underline{u}) = (\lambda \underline{v} \cdot \underline{u})$

$$= \underline{\underline{S}}\,\underline{v} \cdot \underline{u} = \underline{v} \cdot \underline{\underline{S}}^T \underline{u} \qquad \underline{\underline{S}}^T = \underline{\underline{S}}$$

$$= \underline{v} \cdot \underline{\underline{S}}\,\underline{u} = \underline{v} \cdot \omega \underline{u}$$

$$\lambda (\underbrace{\underline{v} \cdot \underline{u}}_{0}) = \omega (\underbrace{\underline{v} \cdot \underline{u}}_{0})$$

$$\underline{v} \perp \underline{u}$$

Spectral decomposition

if $\underline{\underline{S}} = \underline{\underline{S}}^T \rightarrow \{\underline{v}_i\} = \{\underline{e}_i'\}$ $\qquad \underline{v}_i = \underline{e}_i'$

$$\boxed{[\underline{\underline{S}}]' = \sum_{i=1}^{3} \lambda_i \, \underset{\underline{v}_i \otimes \underline{v}_i}{\underline{e}_i' \otimes \underline{e}_i'}} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$$

$$\underline{\underline{I}} = \underline{e}_i' \otimes \underline{e}_i' = \underline{v}_i \otimes \underline{v}_i \qquad (\underline{\underline{\Lambda}})$$

Proof:

$\alpha(\underline{a} \cdot \underline{b}) = (\alpha\underline{a} \cdot \underline{b})$

$$\underline{\underline{S}} = \underline{\underline{S}}\,\underline{\underline{I}} = \underline{\underline{S}}(\underline{v}_i \otimes \underline{v}_i)$$

$$= (\underline{\underline{S}}\,\underline{v}_i \otimes \underline{v}_i) \qquad \text{association}$$

$$= \sum_{i=1}^{3} (\lambda_i \underline{v}_i \otimes \underline{v}_i)$$

$$= \sum_{i=1}^{3} \lambda_i (\underline{v}_i \otimes \underline{v}_i) \qquad \underline{\underline{S}}\,\underline{v}_i = \lambda_i \underline{v}_i$$

$$= \sum_{i=1}^{3} \lambda_i (\underline{e}_i' \otimes \underline{e}_i')$$

Change of basis $\{\underline{e}_i\}$ and $\{\underline{e}_i'\}$
$\uparrow$ eigenvectors

$$\underline{\underline{A}} = \Big[[\underline{e}_1'] \, [\underline{e}_2'] \, [\underline{e}_3'] \Big]$$

$$\Big(= \Big[[\underline{v}_1] \, [\underline{v}_2] \, [\underline{v}_3] \Big] = \underline{\underline{V}} \Big)$$

Marc's lin Alg. world