# Newtonian Fluids

A fluid is incompressible Newtonian if:

1) Reference mass density uniform: $\rho_0(X) = \rho_0$

2) Fluid is incompressible $\nabla_{\underline{x}} \cdot \underline{v} = 0$

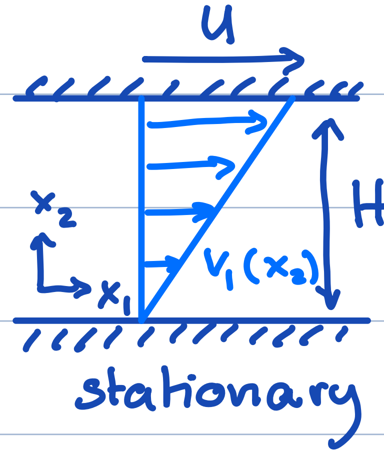
Prop 1 + 2 $\Rightarrow$ $\rho(\underline{x}, t) = \rho_0 > 0$

3) Cauchy stress field is Newtonian

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^h + \underline{\underline{\sigma}}^a$$

hydrost. stress: $\underline{\underline{\sigma}}^h = -p\,\underline{\underline{I}}$

Active stress: $\Rightarrow$ Newton's law of viscosity



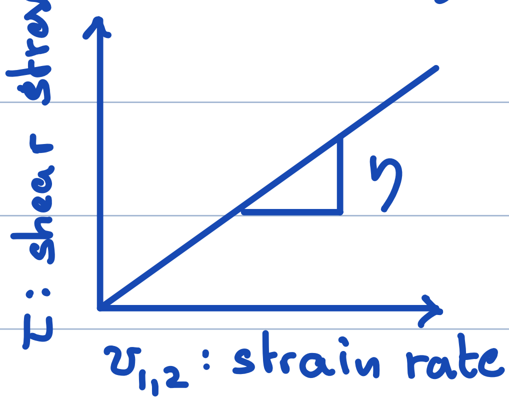

Shear force on top plate

$F = \mu A \frac{U}{H}$ $\quad$ A = area of plate

$\mu$ = dynamic/absolute viscosity

units $\left[\frac{M}{LT}\right]$ e.g. Pa s

shear stress: $\tau = \frac{F}{A}$




$$\boxed{\tau = \mu \frac{\partial v_1}{\partial x_2}}$$ Newton's law of viscosity

## Generalization to 3D

$$\sigma^a_{ij} = \mu \frac{\partial v_i}{\partial x_j} \Rightarrow \underline{\underline{\sigma}}^a = \mu \nabla \underline{v}$$

but $\nabla \underline{v}$ <u>not</u> objective $\Rightarrow \underline{\underline{\sigma}}^a = 2\mu \, \mathrm{sym}(\nabla \underline{v})$

$$= \mu \left( \nabla \underline{v} + \nabla \underline{v}^T \right)$$

$\Rightarrow$ tensor version of classical linear laws

Fourier's law: $\underline{q} = -\kappa \nabla T$ (heat conduction)

Fick's law: $\underline{j} = -D \nabla c$ (species diffusion)

Newton's law: $\underline{\underline{\sigma}} = \mu \, \mathrm{sym}(\nabla \underline{v})$ (momentum diffusion)

## Cauchy stress in Newtonian fluid:

$$\boxed{\underline{\underline{\sigma}} = -p \underline{\underline{I}} + \mu \left( \nabla \underline{v} + \nabla \underline{v}^T \right)}$$

In limit $\mu \to 0$ Newtonian fluid reduces to ideal fluid.

Compare to Representation Thm with $\underline{\underline{E}} = \nabla \underline{v}$:

$$\underline{\underline{G}}(\underline{\underline{E}}) = \lambda \, \mathrm{tr}(\underline{\underline{E}}) \underline{\underline{I}} + 2\mu \, \mathrm{sym}(\underline{\underline{E}})$$

$$\underline{\underline{G}}(\underline{\underline{E}}) = \lambda \cancelto{0}{(\nabla \cdot \underline{v})} \underline{\underline{I}} + \mu \left( \nabla \underline{v} + \nabla \underline{v}^T \right) = \underline{\underline{\sigma}}^a$$

# Navier - Stokes Equations

Setting $\rho = \rho_0$ and $\underline{\underline{\sigma}} = -p\underline{\underline{I}} + \mu(\nabla \underline{v} + \nabla \underline{v}^T)$

we obtain lin. mom. balance

$$\rho \dot{\underline{v}} = \nabla \cdot [-p\underline{\underline{I}} + \mu(\nabla \underline{v} + \nabla \underline{v}^T)] + \rho \underline{g}$$

$$= -\nabla p + \nabla \cdot [\mu(\nabla \underline{v} + \nabla \underline{v}^T)]$$

$$\nabla \cdot (p\underline{\underline{I}}) = (p\,\delta_{ij})_{,j}\,\underline{e}_i = (p_{,j}\,\delta_{ij}\,\underline{e}_j + p\,\cancelto{0}{\delta_{ij,j}}\,\underline{e}_i) = p_{,j}\underline{e}_j = \nabla p$$

assuming $\mu = \text{constant}$ we have

$$\mu \nabla \cdot (\nabla \underline{v} + \nabla \underline{v}^T) = \mu \nabla^2 \underline{v} + \mu \nabla \cdot \nabla \underline{v}^T$$

Last term

$$\nabla \cdot \nabla \underline{v}^T = v_{j,ij}\,\underline{e}_i = v_{j,ji}\,\underline{e}_i = \nabla(\cancelto{0}{\nabla \cdot \underline{v}})$$

$$\Rightarrow \nabla \cdot \underline{\underline{\sigma}} = -\nabla p + \mu \nabla^2 \underline{v}$$

expanding mat. deriv.: $\dot{\underline{v}} = \frac{\partial \underline{v}}{\partial t} + (\nabla \underline{v})\underline{v}$

$$\boxed{\begin{gathered} \rho\left[\frac{\partial \underline{v}}{\partial t} + (\nabla \underline{v})\underline{v}\right] = \mu \nabla^2 \underline{v} - \nabla p + \rho \underline{g} \\ \nabla \cdot \underline{v} = 0 \end{gathered}}$$

Convective form of Navier-Stokes Equation

# Scaling Navier Stokes Equations

$$\rho \frac{\partial \underline{v}}{\partial t} + (\nabla \underline{v})\,\underline{v} = \eta \nabla^2 \underline{v} - \nabla p + \rho g$$

reduced pressure: $\pi = p + \rho g z$

$$-\nabla p + \rho g = -\nabla p - \rho g \hat{z} = -\nabla(p + \rho g z) = -\nabla \pi$$

we have

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + (\nabla \underline{v})\,\underline{v}\right) - \eta \nabla^2 \underline{v} = -\nabla \pi$$

Non-dimensionalize with generic quantities to define standard dimensionless parameters.

- Dependent variables: $\underline{v}$, $\pi$
- Independent variables: $\underline{x}$, $t$
- Parameters: $\rho\left[\frac{M}{L^3}\right]$ $\mu\left[\frac{M}{LT}\right] \rightarrow \nu = \frac{\mu}{\rho}\left[\frac{L^2}{T}\right]$

+ Geometry, BC, IC

Use parameters to scale the variables:

$$\underline{v}' = \frac{\underline{v}}{v_c} \qquad \pi' = \frac{\pi}{\pi_c} \qquad \underline{x}' = \frac{\underline{x}}{x_c} \qquad t' = \frac{t}{t_c}$$

Note: $\underline{v}'$, $\pi'$, $\underline{x}'$ and $t'$ are dimensionless substitute into governing equations

$$\rho \frac{v_c}{t_c} \frac{\partial \underline{v}'}{\partial t'} + \frac{\rho v_c^2}{x_c} (\nabla_x' \underline{v}') \underline{v}' - \frac{\mu v_c}{x_c^2} \nabla_x'^2 \underline{v}' = - \frac{\pi_c}{x_c} \nabla_x' \pi'$$

Choose to scale to accumulation term

$$\frac{\partial \underline{v}'}{\partial t'} + \underbrace{\frac{v_c t_c}{x_c}}_{\Pi_1} (\nabla_x' \underline{v}') \underline{v}' - \underbrace{\frac{\nu t_c}{x_c^2}}_{\Pi_2} \nabla_x'^2 \underline{v}' = - \underbrace{\frac{\pi_c t_c}{x_c \rho_o v_c}}_{\Pi_3} \nabla_x' \pi'$$

where $\nu = \frac{\eta}{\rho}$ kinematic viscosity

$\nu \left[\frac{L^2}{T}\right]$ momentum diffusivity

Three dimensionless groups $\Rightarrow$ define time scale

$\Pi_1 = \frac{v_c t_c}{x_c} = 1 \Rightarrow$ advective scale $t_c = t_A = \frac{x_c}{v_c}$

$\Pi_2 = \frac{\nu t_c}{x_c^2} = 1 \Rightarrow$ diffusive scale $t_c = t_D = \frac{x_c^2}{\nu}$

Use $\Pi_3$ to define pressure scale

$$\Pi_3 = \frac{\pi_c t_c}{x_c \rho_o v_c} = 1 \quad \Rightarrow \quad \pi_c = \frac{x_c \rho_o v_c}{t_c} .$$

Choose a diffusive time scale $t_c = \frac{x_c^2}{\nu}$

Dimensionless N-S equation:

$$\frac{\partial \underline{v}}{\partial t} + \frac{v_c x_c}{\nu}(\nabla'_x \underline{v}')\underline{v}' - \nabla'^2_x \underline{v} = -\nabla'_x \pi'$$

$\Rightarrow$ one remaining dim. less group

$$\boxed{Re = \frac{v_c x_c}{\nu}}$$ Reynolds number

ratio of time scales: $Re = \frac{t_D}{t_A} = \frac{\text{diffusive}}{\text{advective}}$

Hence we have (dropping primes)

$$\boxed{\frac{\partial \underline{v}}{\partial t} + Re\,(\nabla \underline{v})\underline{v} = \nabla^2 \underline{v} - \nabla \pi}$$

$\Rightarrow$ only one governing dimensionless parameter: Reynolds number

For viscous flow of glacier:

$\rho = 10^3 \frac{kg}{m^3}$ $\quad v_c = 100 \frac{m}{yr} \sim 10^{-6} \frac{m}{s}$

$\mu = 10^{14}$ Pas $\quad x_c = 10^2 m$ (thickness)

$$Re = \frac{v_c x_c \rho_0}{\mu_c} = \frac{10^{-6+2+3}}{10^{14}} = 10^{-1-14} = 10^{-15} \ll 1$$

$\Rightarrow$ advective momentum transport is negligible

For $Re = 0$ we have

$$\boxed{\begin{aligned} \frac{\partial \underline{v}}{\partial t} - \nabla^2 \underline{v} &= -\nabla \pi \\ \nabla \cdot \underline{v} &= 0 \end{aligned}}$$

Equations are linear !

But is it worth resolving diffusive timescale ?

$$t_D = \frac{x_c^2 \rho_0}{\mu} = 10^{4+3-14} s = 10^{-7} s$$

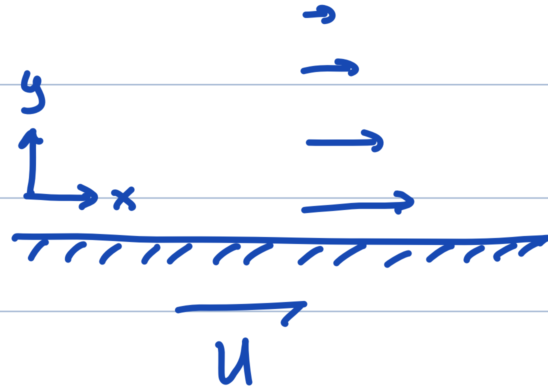
This is very short compared to 100 years of glacier response. Not worth resolving transients.

Can't eliminate transient term because we scaled to it $\Rightarrow$ scale to diffusion term.

$\Rightarrow$ next lecture

# Rayleigh's problem

- Semi-infinite half-space
- Stationary fluid
- Impulsively started plate with velocity $U$.




$$v_c = U \rightarrow Re = \frac{U x_c \rho_0}{\mu} \ll 1 \Rightarrow U \ll \frac{\mu}{x_c \rho_0}$$

But what is $x_c$? Not obvious

Redimensionalize assuming $Re \ll 1$

$$\frac{\partial \underline{v}}{\partial t} - \nu \nabla^2 \underline{v} = -\nabla \pi \quad \& \quad \nabla \cdot \underline{v} = 0 \qquad \underline{v} = \begin{pmatrix} u \\ w \end{pmatrix}$$

Simplify the equations:

Domain is infinite in $x$ but $|\pi| < \infty \Rightarrow \frac{\partial \pi}{\partial x} = 0$

Flow is horizontal: $\underline{v} = \begin{pmatrix} u \\ w \end{pmatrix} \Rightarrow w = 0$

From continuity: $\frac{\partial u}{\partial x} + \cancelto{0}{\frac{\partial w}{\partial y}} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0 \Rightarrow u = u(y)$

$$\nabla^2 \underline{v} = v_{i,jj} \underline{e}_i \qquad i,j \in \{1,2\}$$

$$= \begin{pmatrix} v_{1,11} & v_{1,22} \\ v_{2,11} & v_{2,22} \end{pmatrix} = \begin{pmatrix} \cancelto{0}{u_{xx}} & u_{yy} \\ \cancelto{0}{w_{xx}} & \cancelto{0}{w_{yy}} \end{pmatrix} = \begin{pmatrix} u_{yy} \\ 0 \end{pmatrix}$$

Substituting we have

x-mom.: $\quad \dfrac{\partial u}{\partial t} - \nu \dfrac{\partial^2 u}{\partial x^2} = 0$

y-mom.: $\quad 0 = -\dfrac{\partial \pi}{\partial y}$

$$\Rightarrow \quad \boxed{\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}} \qquad \text{with} \quad u(0,y) = 0 \qquad u(t,0) = U$$

This is identical to heating a semi-infinite rod from the end.

Problem has self-similar solution in

$\eta = \dfrac{y}{\sqrt{4\nu t}}$ and $u(y,t) = U f(\eta)$

where $\sqrt{4\nu t}$ takes role of char. length that depends on $t$.

derivatives: $\dfrac{\partial \eta}{\partial t} = -\dfrac{1}{2}\dfrac{\eta}{t} \qquad \dfrac{\partial \eta}{\partial y} = \dfrac{1}{\sqrt{4\nu t}}$

The derivatives of $u$ transform as:

$$\frac{\partial u}{\partial t} = U \frac{df}{d\eta}\frac{\partial \eta}{\partial t} = -\frac{U}{2}\frac{\eta}{t} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = U \frac{d^2 f}{d\eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 = \frac{U}{4\nu t}\frac{d^2 f}{d\eta^2}$$

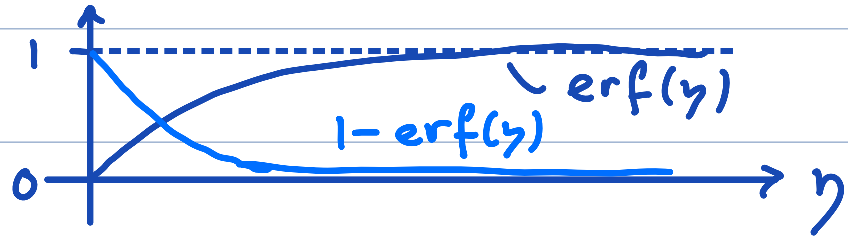
substituting into PDE:

$$\frac{d^2 f}{d\eta^2} + 2\eta \frac{df}{d\eta} = 0 \qquad \text{with} \quad f(\eta = 0) = 1$$

Reduce PDE in $y$ and $t$ to ODE in $\eta$

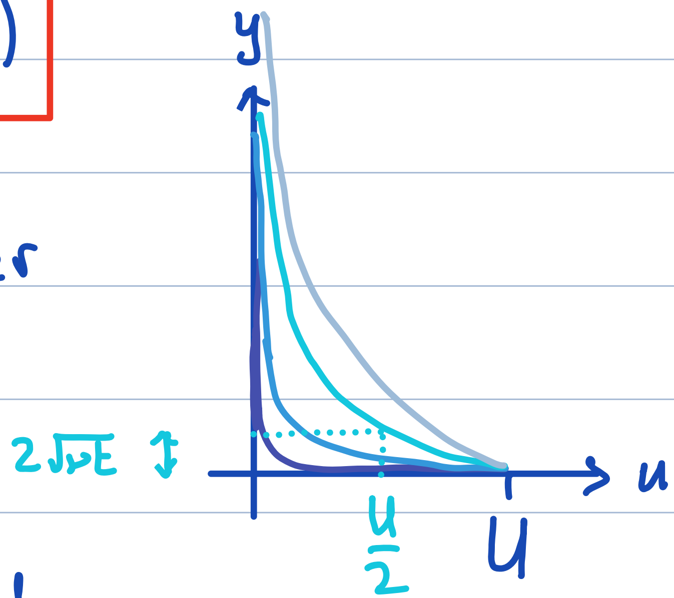
Solution: $f(\eta) = 1 - \mathrm{erf}(\eta)$ (Gauss)

where $\mathrm{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\xi^2} d\xi$ error function




Resubstituting for $f = \frac{u}{U}$ and $\eta = \frac{y}{\sqrt{4\nu t}}$

$$\boxed{u(y,t) = U\left(1 - \mathrm{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)\right)}$$

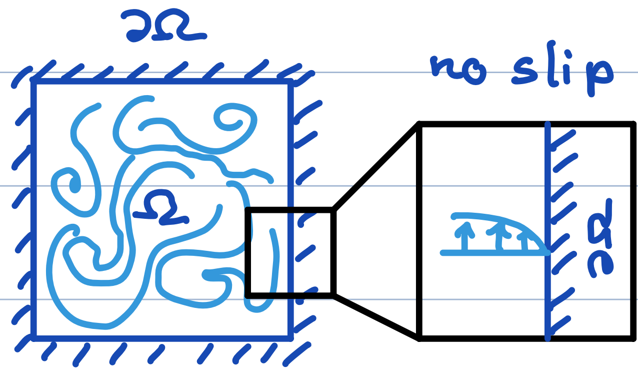



Diffusive boundary layer where momentum added by boundary penetrates into the quiescent fluid.

$\nu = \frac{\mu}{\rho_0}$ is Diffusion coefficient.

# Kinetic Energy of Fluid Motion

Dissipation of kinetic energy in a closed domain with no-slip boundaries.




$\Rightarrow$ Energy is lost by boundary

Change in kinetic energy

$$\frac{d}{dt}K(t) = \int_\Omega \frac{1}{2}\rho \frac{d}{dt}|\underline{v}|^2 \, dV = \int_\Omega \rho \, \dot{\underline{v}} \cdot \underline{v} \, dV$$

from Navier-Stokes Eqns: $\rho_0 \dot{\underline{v}} = \mu \nabla_x^2 \underline{v} - \nabla \pi$

$$\frac{d}{dt}K(t) = \int_\Omega (\mu \nabla^2 \underline{v} - \nabla \pi) \cdot \underline{v} \, dV$$

$$\Rightarrow \frac{dK}{dt} = \mu \int_\Omega \nabla^2 \underline{v} \cdot \underline{v} \, dV - \mu \int_\Omega \nabla \pi \cdot \underline{v} \, dV$$

$\Rightarrow$ show second term is zero

$$\nabla \cdot (\pi \underline{v}) = \underline{v} \cdot \nabla \pi + \pi \, \cancel{\nabla \cdot \underline{v}}^{\,0}$$

$$\int_\Omega \nabla\pi\cdot\underline{v}\,dV = \int_\Omega \nabla\cdot(\pi v)\,dV = \int_{\partial\Omega} (\pi \cancel{\underline{v}})\cdot\hat{\underline{n}}\,dA \xrightarrow{} 0$$

$\underline{v} = 0$ on $\partial\Omega$

$$\Rightarrow \frac{dK}{dt} = \mu \int_\Omega \nabla^2\underline{v}\cdot\underline{v}\,dV = ?$$

$$\nabla^2\underline{v}\cdot\underline{v} = v_{i,jj}\cdot v_i = (v_{i,j}v_i)_{,j} - v_{i,j}v_{i,j}$$
$$= \nabla\cdot(\nabla^T\underline{v}\,\underline{v}) - \nabla\underline{v} : \nabla\underline{v}$$

$$\frac{dK}{dt} = \mu\int_\Omega \nabla\cdot(\nabla^T\underline{v}\,\underline{v})\,dV - \mu\int_\Omega \nabla\underline{v}:\nabla\underline{v}\,dV$$
$$= \mu\int_{\partial\Omega} \cancel{(\nabla\underline{v}^T\underline{v})}\cdot\hat{\underline{n}}\,dV - \mu\int_\Omega \nabla\underline{v}:\nabla\underline{v}\,dV$$

$$\frac{dK}{dt} = -\mu\int_\Omega \nabla\underline{v}:\nabla\underline{v}\,dV$$

Poincaré Inequality

$\|\underline{v}\|_\Omega \leq \lambda\|\nabla\underline{v}\|_\Omega$ for $\underline{v}=0$ $\partial\Omega$ $\lambda > 0$

using standard inner product

$$\boxed{\int_\Omega |\underline{v}|^2\,dV \leq \lambda\int_\Omega \nabla\underline{v}:\nabla\underline{v}\,dV}$$

Notice $\lambda$ has units of $L^2$ and scales with area of $\Omega$.

$$\frac{dK}{dt} \le -\mu\lambda \int_\Omega |\underline{v}|^2 dV = -\frac{\mu\lambda}{\rho_0} \underbrace{\int_\Omega \rho |\underline{v}|^2 dV}_{K}$$

$$\Rightarrow \quad \boxed{\frac{dK}{dt} \le -\frac{\mu\lambda}{\rho_0} K}$$

Solve by seperation of parts

$$\frac{dK}{K} \le -\frac{2\mu}{\rho_0\lambda} dt = -\alpha\, dt$$

$$\ln k \le -\alpha t + c_0$$

$$k \le c_1 e^{-\alpha t}$$

Initial condition $\quad K(0) \le c_1 = K_0$

$$\boxed{K(t) \le K_0 \, e^{-\frac{\nu}{\lambda} t}}$$

$\nu = \frac{\mu}{\rho}$ mom. diffusivity

$\lambda =$ geometric factor