Momentum diffusion

Remember where terms come from

lin. mom. bal.: $\rho \dot{\underline{v}} = \nabla \cdot \underline{\underline{\sigma}} + \rho g$

Newtonian $\quad \underline{\underline{\sigma}} = -p\underline{\underline{I}} + \mu(\nabla \underline{v} + \nabla \underline{v}^T)$

Navier-Stokes: $\frac{\partial v}{\partial t} + \underline{v} \cdot \nabla \underline{v} = \nu \nabla^2 \underline{v} - \frac{1}{\rho} \nabla \pi$

kinematic viscosity: $\nu = \frac{\mu}{\rho} \quad \left[\frac{m^2}{s}\right]$

$\mu = Pa\,s = \frac{kg}{m\,s} \quad \rho = \frac{kg}{m^3}$

$\Rightarrow$ units of diffusivity

Today we'll show that ν characterizes momentum diffusion !

Scaled N-S equations:

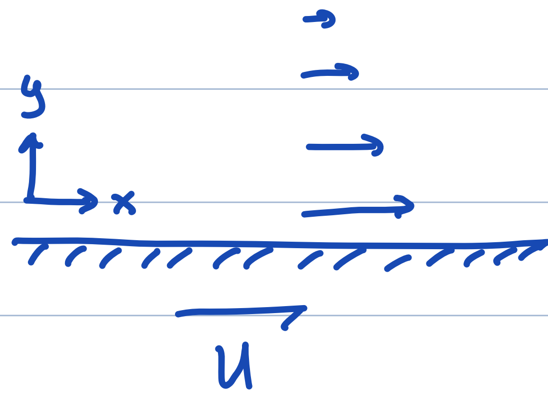
$$\frac{\partial v}{\partial t} + Re\, \underline{v} \cdot \nabla \underline{v} = \nabla^2 \underline{v} - \nabla \pi$$

$Re = \frac{v_c x_c}{\nu}$ Reynolds number

Today $Re \ll 1$

Rayleigh's problem

- Semi-infinite half-space
- Stationary fluid
- Impulsively started plate with velocity U.



$$v_c = U \rightarrow Re = \frac{U x_c \rho_0}{\mu} \ll 1 \Rightarrow U \ll \frac{\mu}{x_c \rho_0}$$

But what is x_c? Not obvious

Redimensonalize assuming $Re \ll 1$

$$\frac{\partial \underline{v}}{\partial t} - \nu \nabla^2 \underline{v} = -\nabla \pi \quad \& \quad \nabla \cdot \underline{v} = 0 \qquad \underline{v} = \begin{pmatrix} u \\ w \end{pmatrix}$$

Simplify the equations:

Domain is infinite in x but $|\pi| < \infty \Rightarrow \frac{\partial \pi}{\partial x} = 0$

Flow is horizontal: $\underline{v} = \begin{pmatrix} u \\ w \end{pmatrix} \Rightarrow w = 0$

From continuity: $\frac{\partial u}{\partial x} + \cancelto{0}{\frac{\partial w}{\partial y}} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0 \Rightarrow u = u(y)$

$$\nabla^2 \underline{v} = v_{i,jj} \underline{e}_i \qquad i,j \in \{1,2\}$$

$$= \begin{pmatrix} v_{1,11} & v_{1,22} \\ v_{2,11} & v_{2,22} \end{pmatrix} = \begin{pmatrix} \cancelto{0}{u_{xx}} & u_{yy} \\ \cancelto{0}{w_{xx}} & \cancelto{0}{w_{yy}} \end{pmatrix} = \begin{pmatrix} u_{yy} \\ 0 \end{pmatrix}$$

Substituting we have

x-mom.: $\frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial x^2} = 0$

y-mom.: $0 = -\frac{\partial \pi}{\partial y} \Rightarrow \nabla \pi = \underline{0}$

$$\Rightarrow \boxed{\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}} \quad \text{with} \quad u(0,y)=0 \quad u(t,0)=U$$

This is identical to heating a semi-infinite rod from the end.

Problem has self-similar solution in

$\eta = \frac{y}{\sqrt{4\nu t}}$ and $u(y,t) = U f(\eta)$

where $\sqrt{4\nu t}$ takes role of char. length that depends on t.

derivatives: $\frac{\partial \eta}{\partial t} = -\frac{1}{2}\frac{\eta}{t}$ $\quad \frac{\partial \eta}{\partial y} = \frac{1}{\sqrt{4\nu t}}$

The derivatives of u transform as:

$$\frac{\partial u}{\partial t} = U \frac{df}{d\eta}\frac{\partial \eta}{\partial t} = -\frac{U}{2}\frac{\eta}{t} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = U \frac{d^2 f}{d\eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 = \frac{U}{4\nu t}\frac{d^2 f}{d\eta^2}$$

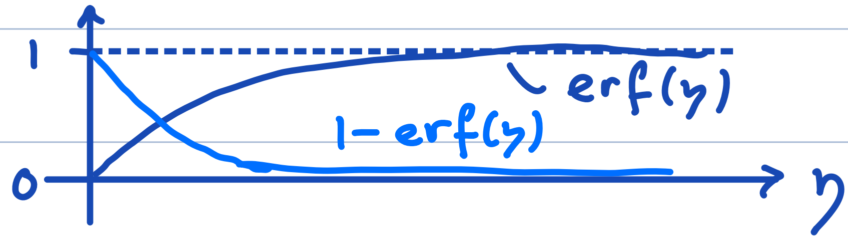
substituting into PDE:

$$\frac{d^2 f}{d\eta^2} + 2\eta \frac{df}{d\eta} = 0 \quad \text{with} \quad f(\eta=0)=1$$

Reduce PDE in y and t to ODE in η

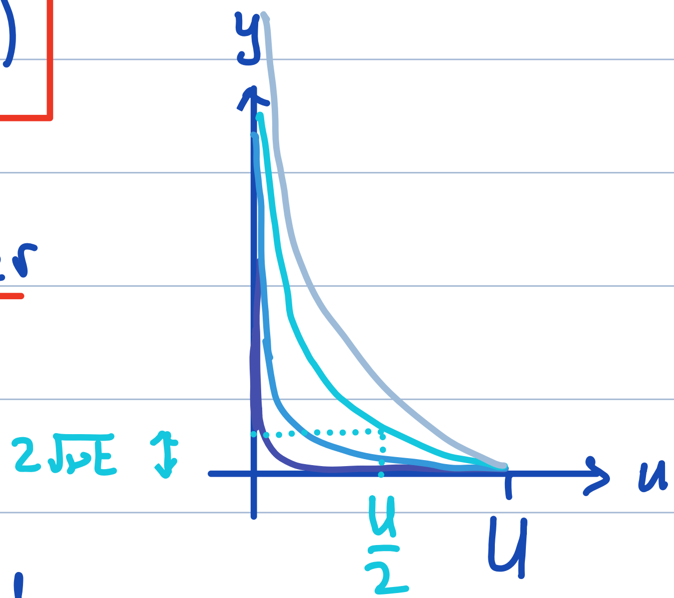
Solution : $f(\eta) = 1 - \mathrm{erf}(\eta)$ (Gauss)

where $\mathrm{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\xi^2} d\xi$ error function



Resubstituting for $f = \frac{u}{U}$ and $\eta = \frac{y}{\sqrt{4\nu t}}$

$$\boxed{u(y,t) = U\left(1 - \mathrm{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)\right)}$$



Diffusive boundary layer where momentum added by boundary penetrates into the quiescent fluid.

$\nu = \frac{\mu}{\rho_0}$ is Diffusion coefficient.

$\Rightarrow$ shear stress causes momentum diffusion lin. mom. bal.: $\rho \dot{\underline{v}} = \nabla \cdot \underline{\underline{\sigma}} + \rho g$

rewrite in conservative form:

$$\frac{\partial}{\partial t}(\rho \underline{v}) + \nabla \cdot [\underbrace{\rho \underline{v} \otimes \underline{v}}_{\text{adv. flux}} - \underbrace{\underline{\underline{\sigma}}}_{\text{diffusive flux}}] = \rho g$$

Energy dissipation in viscous boundary layer

stress power: $\phi = \underline{\underline{\sigma}} : \underline{\underline{d}}$ $\qquad$ $\underline{\underline{\sigma}} = -p\underline{\underline{I}} + 2\mu \underline{\underline{d}}$

$$= -p\underline{\underline{I}} : \underline{\underline{d}} + 2\mu\, \underline{\underline{d}} : \underline{\underline{d}}$$

$$\underline{\underline{I}} : \underline{\underline{d}} = \delta_{ij} \frac{1}{2}(v_{i,j} + v_{j,i}) = v_{i,i} = \nabla \cdot \underline{v} = 0 \text{ incompr.}$$

$\Rightarrow$ $\boxed{\phi = 2\mu\, \underline{\underline{d}} : \underline{\underline{d}}}$ stress power in incompressible fluid

Example: $u(y,t) = U\left[1 - \operatorname{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)\right]$

2D: $\nabla \underline{v} = \begin{bmatrix} u_{,x} & u_{,y} \\ w_{,x} & w_{,y} \end{bmatrix} = \begin{bmatrix} 0 & u_{,y} \\ 0 & 0 \end{bmatrix}$

$$\underline{\underline{d}} = \frac{1}{2}\begin{bmatrix} 0 & u_{,y} \\ u_{,y} & 0 \end{bmatrix}$$

stress power: $\phi = 2\mu \, \underline{\underline{d}} : \underline{\underline{d}} = 2\mu \, d_{ij} d_{ij}$

$$= 2\mu\left[\tfrac{1}{4}(u_{,y})^2 + \tfrac{1}{4}(u_{,y})^2\right]$$

$$\boxed{\phi = \mu (u_{,y})^2}$$

$$\frac{\partial u}{\partial y} = -U \frac{d}{dy} \operatorname{erf}\left(\frac{y}{\sqrt{4\nu t}}\right) = -U \frac{d}{d\eta} \operatorname{erf}(\eta) \frac{d\eta}{dy}$$

$$\operatorname{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\xi^2} d\xi \Rightarrow \frac{d}{d\eta} \operatorname{erf}(\eta) = \frac{2}{\sqrt{\pi}} e^{-\eta^2}$$

$$\frac{d\eta}{dy} = \frac{1}{\sqrt{4\nu t}}$$

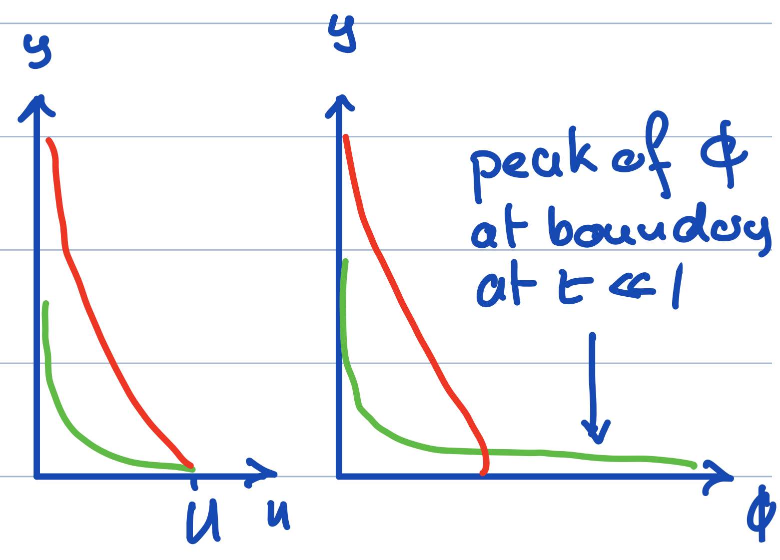
$$\Rightarrow \frac{\partial u}{\partial t} = -\frac{U}{\sqrt{\pi \nu t}} e^{-\frac{y^2}{4\nu t}}$$

Energy dissipation:

$$\phi = \mu \left(\frac{\partial u}{\partial y}\right)^2 = \frac{\mu U^2}{\pi \underset{\substack{\uparrow \\ \mu/\rho}}{\nu} t} e^{-\frac{y^2}{4\nu t}} = \frac{\rho U^2}{\pi t} e^{-\frac{y^2}{4\nu t}}$$

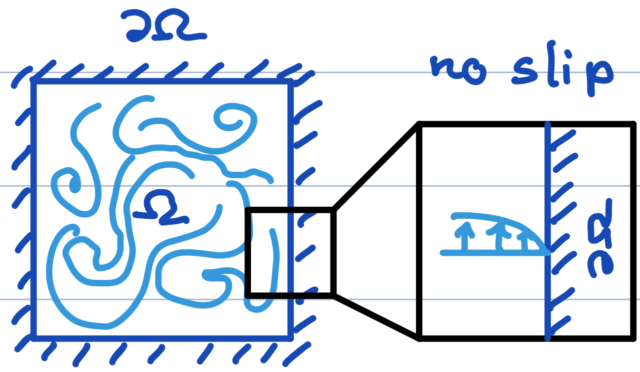
$$\boxed{\phi(y,t) = \frac{\rho U^2}{\pi t} e^{-\frac{\rho y^2}{4\mu t}}}$$

Gaussian



Kinetic Energy of Fluid Motion

Dissipation of kinetic energy in a closed domain with no-slip boundaries.



$\Rightarrow$ Energy is lost by boundary

Change in kinetic energy

$$\frac{d}{dt} K(t) = \int_\Omega \frac{1}{2}\rho \frac{d}{dt} |\underline{v}|^2 \, dV = \int_\Omega \rho \, \dot{\underline{v}} \cdot \underline{v} \, dV$$

from Navier-Stokes Eqns: $\rho_0 \dot{\underline{v}} = \mu \nabla_x^2 \underline{v} - \nabla \pi$

$$\frac{d}{dt} K(t) = \int_\Omega (\mu \nabla^2 \underline{v} - \nabla\pi) \cdot \underline{v} \, dV$$

$$\Rightarrow \frac{dK}{dt} = \mu \int_\Omega \nabla^2 \underline{v} \cdot \underline{v} \, dV - \mu \int_\Omega \nabla\pi \cdot \underline{v} \, dV$$

$\Rightarrow$ show second term is zero

$$\nabla \cdot (\pi \underline{v}) = \underline{v} \cdot \nabla\pi + \pi \cancelto{0}{\nabla \cdot \underline{v}}$$

$$\int_\Omega \nabla\pi \cdot \underline{v}\, dV = \int_\Omega \nabla\cdot(\pi \underline{v})\, dV = \int_{\partial\Omega} \cancel{(\pi \underline{v})}^{\,0} \cdot \hat{\underline{n}}\, dA$$

$\underline{v} = 0$ on $\partial\Omega$

$$\Rightarrow \frac{dK}{dt} = \mu \int_\Omega \nabla^2 \underline{v} \cdot \underline{v}\, dV = \,?$$

$$\nabla^2 \underline{v} \cdot \underline{v} = v_{i,jj} \cdot v_i = (v_{i,j} v_i)_{,j} - v_{i,j} v_{i,j}$$
$$= \nabla\cdot(\nabla^T \underline{v}\, \underline{v}) - \nabla \underline{v} : \nabla \underline{v}$$

$$\frac{dK}{dt} = \mu \int_\Omega \nabla\cdot(\nabla^T \underline{v}\, \underline{v})\, dV - \mu \int_\Omega \nabla \underline{v} : \nabla \underline{v}\, dV$$

$$= \mu \int_{\partial\Omega} \cancel{(\nabla^T \underline{v}\, \underline{v})}^{\,0} \cdot \hat{\underline{n}}\, dV - \mu \int_\Omega \nabla \underline{v} : \nabla \underline{v}\, dV$$

$$\frac{dK}{dt} = -\mu \int_\Omega \nabla \underline{v} : \nabla \underline{v}\, dV$$

Poincaré Inequality

$\|\underline{v}\|_\Omega \leq \lambda \|\nabla \underline{v}\|_\Omega$ for $\underline{v} = 0$ $\partial\Omega$ $\lambda > 0$

using standard inner product

$$\boxed{\int_\Omega |\underline{v}|^2\, dV \leq \lambda \int_\Omega \nabla \underline{v} : \nabla \underline{v}\, dV}$$

Notice λ has units of L^2 and scales with area of Ω.

$$\frac{dK}{dt} \leq -\mu\lambda \int_{\Omega} |\underline{v}|^2 dV = -\frac{\mu\lambda}{\rho_0} \underbrace{\int_{\Omega} \rho |\underline{v}|^2 dV}_{K}$$

$$\Rightarrow \quad \boxed{\frac{dK}{dt} \leq -\frac{\mu\lambda}{\rho_0} K}$$

Solve by separation of parts

$$\frac{dK}{K} \leq -\frac{2\mu}{\rho_0\lambda} dt = -\alpha\, dt$$

$$\ln k \leq -\alpha t + c_0$$

$$k \leq c_1 e^{-\alpha t}$$

Initial condition $\quad K(0) \leq c_1 = K_0$

$$\boxed{K(t) \leq K_0\, e^{-\frac{\nu}{\lambda} t}}$$

$\nu = \frac{\mu}{\rho}$ mom. diffusivity

$\lambda =$ geometric factor