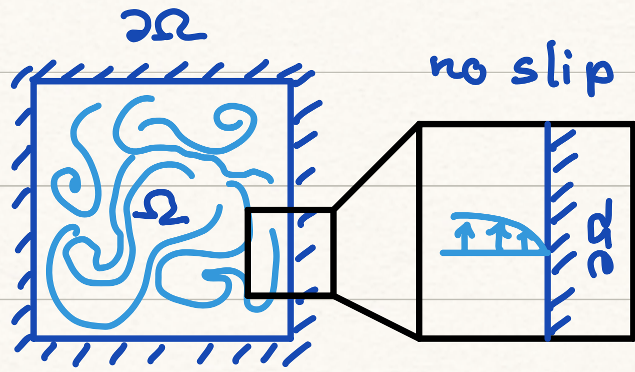
Kinetic Energy of Fluid Motion

Dissipation of kinetic energy in a closed domain with no-slip boundaries.



$\Rightarrow$ Energy is lost by boundary

Change in kinetic energy

$$\frac{d}{dt}K(t) = \int_\Omega \frac{1}{2}\rho \frac{d}{dt}|\underline{v}|^2 \, dV = \int_\Omega \rho \, \dot{\underline{v}} \cdot \underline{v} \, dV$$

from Navier-Stokes Eqns: $\rho_0 \dot{\underline{v}} = \mu \nabla_x^2 \underline{v} - \nabla \pi$

$$\frac{d}{dt}K(t) = \int_\Omega \left(\mu \nabla^2 \underline{v} - \nabla \pi\right) \cdot \underline{v} \, dV$$

$$\Rightarrow \frac{dK}{dt} = \mu \int_\Omega \nabla^2 \underline{v} \cdot \underline{v} \, dV - \mu \int_\Omega \nabla \pi \cdot \underline{v} \, dV$$

$\Rightarrow$ show second term is zero

$$\nabla \cdot (\pi \underline{v}) = \underline{v} \cdot \nabla \pi + \pi \, \cancelto{0}{\nabla \cdot \underline{v}}$$

$$\int_\Omega \nabla \pi \cdot \underline{v}\, dV = \int_\Omega \nabla\cdot(\pi \underline{v})\, dV = \int_{\partial\Omega} \overset{0}{(\pi \cancel{\underline{v}})} \cdot \underline{\hat{n}}\, dA$$

$\underline{v} = 0$ on $\partial\Omega$

$$\Rightarrow \frac{dK}{dt} = \mu \int_\Omega \nabla^2 \underline{v} \cdot \underline{v}\, dV = \; ?$$

$$\nabla^2 \underline{v} \cdot \underline{v} = v_{i,jj} \cdot v_i = (v_{i,j} v_i)_{,j} - v_{i,j} v_{i,j}$$

$$= \nabla\cdot(\nabla^T \underline{v}\, \underline{v}) - \nabla \underline{v} : \nabla \underline{v}$$

$$\frac{dK}{dt} = \mu \int_\Omega \nabla\cdot(\nabla^T \underline{v}\, \underline{v})\, dV - \mu \int_\Omega \nabla \underline{v} : \nabla \underline{v}\, dV$$

$$= \mu \int_{\partial\Omega} \overset{0}{\cancel{(\nabla^T \underline{v}\, \underline{v})}} \cdot \underline{\hat{n}}\, dV - \mu \int_\Omega \nabla \underline{v} : \nabla \underline{v}\, dV$$

$$\frac{dK}{dt} = -\mu \int_\Omega \nabla \underline{v} : \nabla \underline{v}\, dV$$

Poincaré Inequality

$\|\underline{v}\|_\Omega \leq \lambda \|\nabla \underline{v}\|_\Omega$ for $\underline{v} = 0$ $\partial\Omega$ $\lambda > 0$

using standard inner product

$$\boxed{\int_\Omega |\underline{v}|^2\, dV \leq \lambda \int_\Omega \nabla \underline{v} : \nabla \underline{v}\, dV}$$

Notice λ has units of L^2 and scales with area of Ω.

$$\frac{dK}{dt} \leq -\mu\lambda \int_\Omega |\underline{v}|^2 dV = -\frac{\mu\lambda}{\rho_0} \underbrace{\int_\Omega \rho |\underline{v}|^2 dV}_{K}$$

$$\Rightarrow \quad \boxed{\frac{dK}{dt} \leq -\frac{\mu\lambda}{\rho_0} K}$$

Solve by seperation of parts

$$\frac{dK}{K} \leq -\frac{2\mu}{\rho_0\lambda} dt = -\alpha\, dt$$

$$\ln K \leq -\alpha t + c_0$$

$$K \leq c_1 e^{-\alpha t}$$

Initial condition $\quad K(0) \leq c_1 = K_0$

$$\boxed{K(t) \leq K_0 e^{-\frac{\nu}{\lambda} t}}$$

$\nu = \frac{\mu}{\rho}$ mom. diffusivity

$\lambda =$ geometric factor