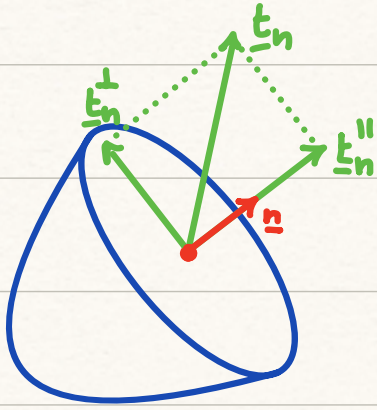


Normal and Shear Stresses



Consider an arbitrary surface in B with normal $\underline{n}$.

$$\underline{P}'' = \underline{n} \otimes \underline{n}$$

$$\underline{P}^+ = \underline{I} - \underline{n} \otimes \underline{n}$$

normal traction: $\underline{t}'' = (\underline{t} \cdot \underline{n}) \underline{n} = \underline{P}'' \underline{t} = (\underline{n} \otimes \underline{n}) \underline{t}$

shear traction: $\underline{t}^+ = \underline{t} - \underline{t}'' = \underline{P}^+ \underline{t} = (\underline{I} - \underline{n} \otimes \underline{n}) \underline{t} = -\underline{t} \times \underline{n} \times \underline{n}$

The magnitudes of these tractions are

normal stress: $\sigma_n = |\underline{t}''|$

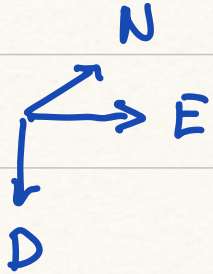
shear stress: $\tau = |\underline{t}^+|$

$$\sigma_n > 0 \rightarrow \text{compressive stress}$$

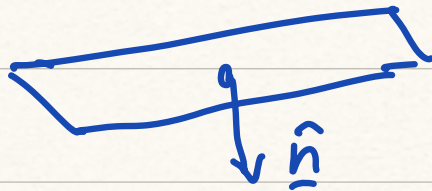
$$\sigma_n < 0 \rightarrow \text{tensile stress}$$

Example: North. Hikurangi subduction zone
(Emma Parker) IODP Expedition 372 & 375

Bore hole U1518B $\rightarrow$ Papaku Fault



Fault normal: $\hat{n} = \begin{bmatrix} 0.104 \\ -0.18 \\ -0.978 \end{bmatrix}$ $\begin{matrix} N \\ E \\ D \end{matrix}$



Stress on fault: $\underline{\underline{\sigma}} = \begin{bmatrix} 26 & 0.5 & 0 \\ 0.5 & 17 & 0 \\ 0 & 0 & 11 \end{bmatrix}$ MPa

traction on fault:

$$\underline{t} = \underline{\underline{\sigma}} \underline{n} = \begin{bmatrix} 26 & 0.5 & 0 \\ 0.5 & 17 & 0 \\ 0 & 0 & 11 \end{bmatrix} \begin{bmatrix} 0.104 \\ -0.18 \\ -0.978 \end{bmatrix} = \begin{bmatrix} 2.61 \\ -3.01 \\ -10.76 \end{bmatrix} \text{ MPa}$$

$$\underline{t}'' = (\underline{n} \cdot \underline{t}) \underline{n} = \begin{bmatrix} 1.18 \\ -2.04 \\ -11.09 \end{bmatrix} \text{ MPa} \quad \sigma_n = |\underline{t}''| = 11.34 \text{ MPa}$$

$$\underline{t}^\perp = \underline{t} - \underline{t}'' = \begin{bmatrix} 1.44 \\ -0.97 \\ 0.33 \end{bmatrix} \text{ MPa} \quad \tau = |\underline{t}^\perp| = 1.76 \text{ MPa}$$

Principal Stresses

Find planes with max. & min. normal stress.

$$\sigma_n = |t''| = n \cdot t = n \cdot \frac{1}{n} = 1$$

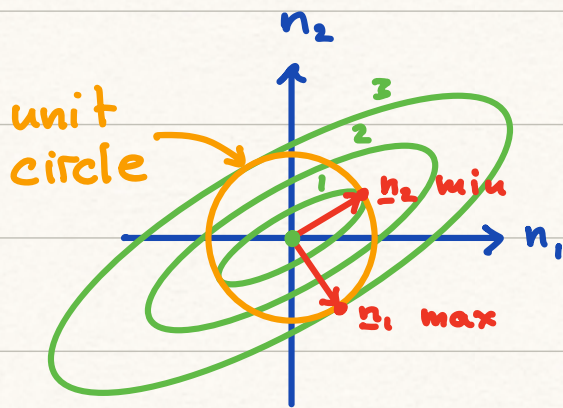
n = normal

$$\hat{\sigma}_n(\underline{n}) = \underline{n} \cdot \underline{\underline{\sigma}} \underline{n}$$

find $\max(\sigma_n)$ & $\min(\sigma_n)$

constraint: $|\underline{n}| = \underline{n} \cdot \underline{n} = 1$

⇒ constrained optimization



$\sigma_n(n) = n \cdot \underline{\sigma} n$ is quadratic

Note: Contours are ellipses if eigenvalues of $\underline{\underline{S}}$ are positive!

Lagrange multiplier method

$$\mathcal{L}(\underline{n}, \lambda) = \underline{n} \cdot \underline{\sigma} \underline{n} - \lambda (\underline{n} \cdot \underline{n} - 1)$$

$$\mathcal{L}(n_i, \lambda) = n_i \sigma_{ij} n_j - \lambda (n_i n_i - 1)$$

function

constraint

Lagrange multipliert

stationary points of $\mathcal{L}(\underline{n}, \lambda)$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \quad \frac{\partial \mathcal{L}}{\partial n_k} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = n_i n_i - 1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial n_k} = \sigma_{ij} (n_{i,k} n_j + n_i n_{j,k}) - \lambda (2 n_i n_{i,k}) = 0$$

$$\text{where } n_{i,k} = \delta_{ik} \quad n_{j,k} = \delta_{jk}$$

$$= \sigma_{ij} (\delta_{ik} n_j + \delta_{jk} n_i) - \lambda (2 n_i \delta_{ik})$$

$$= \sigma_{kj} n_j + \sigma_{ik} n_i - 2 \lambda n_k \quad \sigma_{ik} = \sigma_{ki} \quad (\underline{\sigma}^T = \underline{\sigma})$$

$$= \sigma_{kj} n_j + \sigma_{ki} n_i - 2 \lambda n_k \quad \text{rename } i \rightarrow j$$

$$= \sigma_{kj} n_j + \sigma_{kj} n_j - 2 \lambda n_k$$

$$= 2 (\sigma_{kj} n_j - \lambda n_k) = 0$$

$$n_k = \sigma_{kj} n_j$$

$$= (\sigma_{kj} - \lambda \delta_{kj}) n_j$$

remove free index

$$= (\sigma_{kj} - \lambda \delta_{kj}) n_j = 0$$

$$= (\underline{\sigma} - \lambda \underline{I}) \underline{n}$$

Condition for stationary point

$$(\underline{\sigma} - \lambda \underline{I}) \underline{n} = \underline{0} \quad \text{and} \quad |\underline{n}| = 1$$

$\Rightarrow$ eigenvalue problem!

What does it mean?

$$\underline{n} \cdot (\underline{\underline{\sigma}} - \lambda \underline{\underline{I}}) \underline{n} = 0$$

$$\underbrace{\underline{n} \cdot \underline{\underline{\sigma}} \underline{n}}_{\sigma_n} - \lambda \underbrace{\underline{n} \cdot \underline{n}}_1 = 0 \quad \Rightarrow \quad \boxed{\sigma_n = \lambda}$$

eigen values $\rightarrow$ normal stresses

note: $\lambda_i, \underline{n}_i$ three eigen values & vectors

change notation: $\lambda_i = \sigma_i \quad \underline{n}_i = \underline{v}_i$

$$\Rightarrow (\underline{\underline{\sigma}} - \sigma_i \underline{\underline{I}}) \underline{v}_i = 0$$

$\sigma_1 > \sigma_2 > \sigma_3$: principal normal stresses
 $\uparrow \quad \quad \quad \uparrow$
max min

$\underline{v}_i$: principal directions of $\underline{\underline{\sigma}}$

$\underline{\underline{\sigma}}^T = \underline{\underline{\sigma}} \Rightarrow \{ \underline{v}_i \}$ form principal frame
(orthonormal)

always choose right handed

Cauchy stress in principal frame

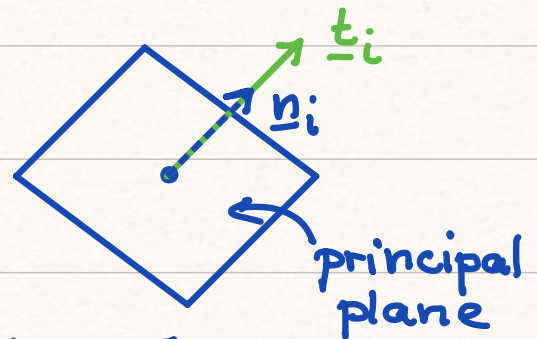
$$\underline{\underline{\sigma}} = \sum_{i=1}^3 \sigma_i \underline{v}_i \otimes \underline{v}_i = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

Compare to $\{\underline{e}_i\}$

$$\underline{\underline{\sigma}} = \sigma_{ij} \underline{e}_i \otimes \underline{e}_j$$

Traction on principal directions $\underline{n} = \underline{v}_i$

$$\begin{aligned} \underline{t} &= \underline{\underline{\sigma}} \underline{v}_i = \sum_{k=1}^3 \sigma_k (\underline{v}_k \otimes \underline{v}_k) \underline{v}_i \\ &= \sum_{k=1}^3 \sigma_k (\underline{v}_k \cdot \underline{v}_i) \underline{v}_k \\ &= \sigma_k \delta_{ki} \underline{v}_k \\ &= \sigma_i \underline{v}_i \quad \text{no summation!} \end{aligned}$$



$$\underline{t}'' = (\underline{t} \cdot \underline{v}_i) \underline{v}_i \quad \text{no summation}$$

$$= \sigma_i (\underline{v}_i \cdot \underline{v}_i) \underline{v}_i = \sigma_i \underline{v}_i = \underline{t}$$

$$\Rightarrow \sigma_n = \sigma_i$$

$$\underline{t}^\tau = \underline{t} - \underline{t}'' = \underline{0} \quad \Rightarrow \quad \tau = 0$$

$\Rightarrow$ no shear stress on principal planes