Orthogonal tensors

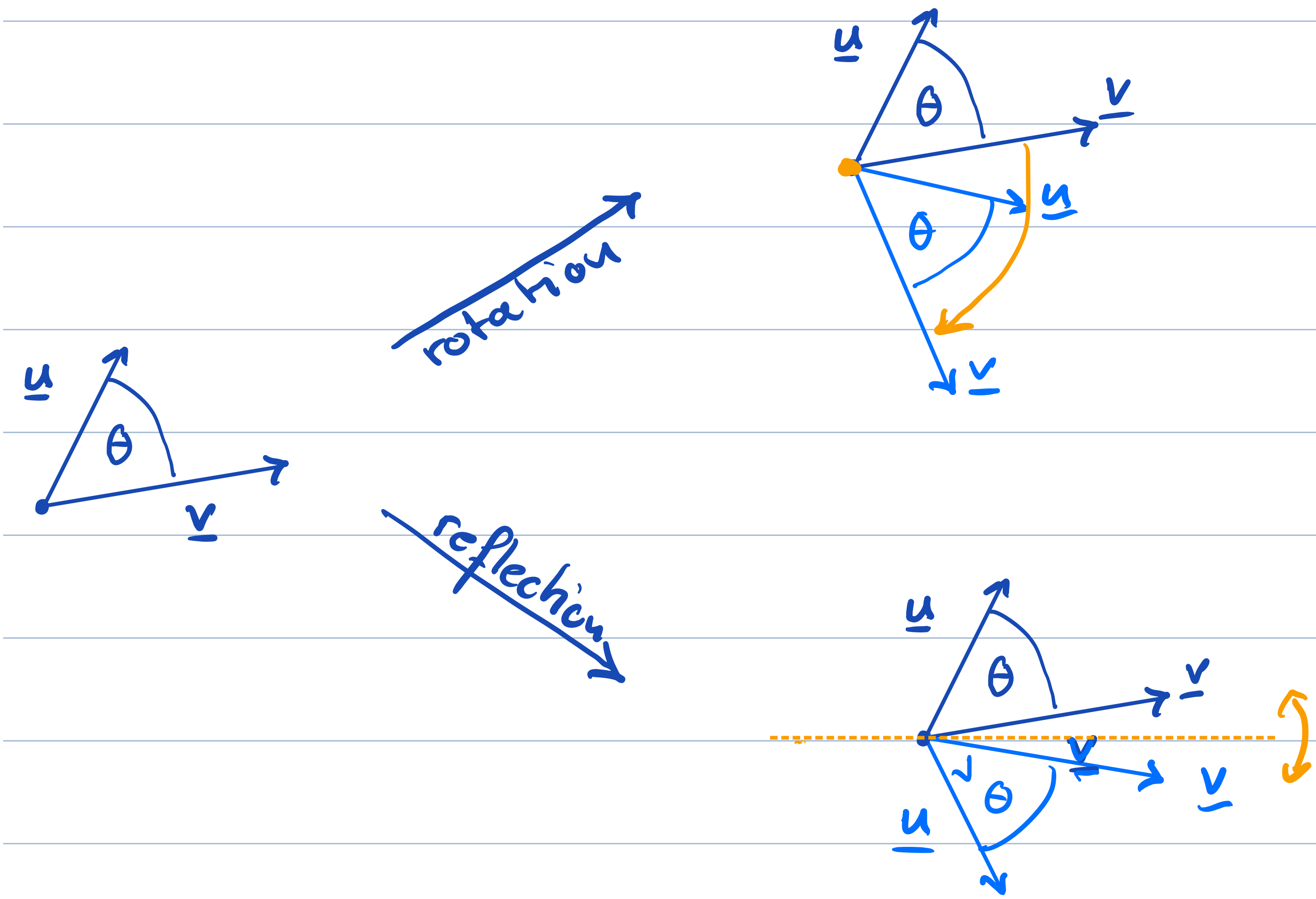
An orthogonal tensor $\underline{\underline{Q}} \in \mathcal{V}^2$ is a linear transformation satisfying

$$\boxed{\underline{\underline{Q}}\,\underline{u} \cdot \underline{\underline{Q}}\,\underline{v} = \underline{u} \cdot \underline{v}} \qquad \text{for all } \underline{u}, \underline{v} \in \mathcal{V}$$

$$\underline{u} \cdot \underline{v} = |\underline{u}|\,|\underline{v}| \cos\theta$$

$\Rightarrow$ preserves length & angle

only two possible operations:



Properties of orthogonal matrices:

$$\boxed{\begin{gathered}\underline{\underline{Q}}^T = \underline{\underline{Q}}^{-1} \\ \underline{\underline{Q}}^T\underline{\underline{Q}} = \underline{\underline{Q}}\,\underline{\underline{Q}}^T = \underline{\underline{I}} \\ \det(\underline{\underline{Q}}) = \pm 1\end{gathered}}$$

Show last: $\quad 1 = \det(\underline{\underline{I}}) = \det(\underline{\underline{Q}}^T\underline{\underline{Q}})$

$$= \det(\underline{\underline{Q}}^T)\det(\underline{\underline{Q}}) = \det(\underline{\underline{Q}})^2$$

$$\Rightarrow \quad \det(\underline{\underline{Q}}) = \pm 1$$

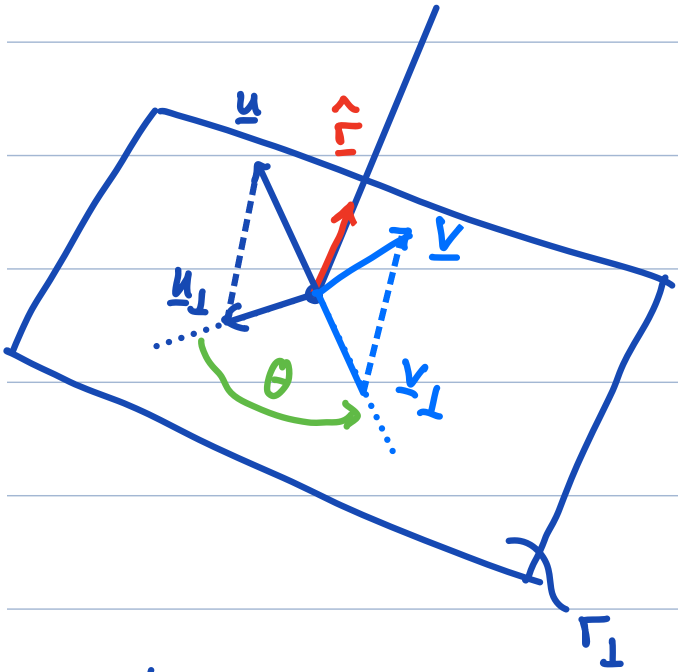
If $\det(\underline{\underline{Q}}) = 1 \Rightarrow$ rotation

$\det(\underline{\underline{Q}}) = -1 \Rightarrow$ reflection

In mechanics we are mostly concerned with rotations.

Rotation Matrices

$\underline{v} = Q(\hat{\underline{r}}, \theta)\, \underline{u}$



$\hat{\underline{r}}$ = axis of rotation

$r_\perp$ = plane $\perp$ to $\hat{\underline{r}}$

θ = counter clock wise angle

$\underline{u} = \underline{u}_\parallel + \underline{u}_\perp$

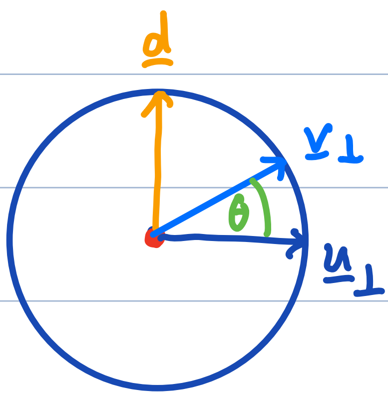
$\underline{v} = \underline{v}_\parallel + \underline{v}_\perp$

$\underline{v}_\parallel = \underline{u}_\parallel = (\underline{u} \cdot \hat{\underline{r}})\hat{\underline{r}} = (\hat{\underline{r}} \otimes \hat{\underline{r}})\, \underline{u}$

$\underline{u}_\perp = (\underline{\underline{I}} - \hat{\underline{r}} \otimes \hat{\underline{r}})\, \underline{u}$

What is $\underline{v}_\perp$?

looking onto $r_\perp$



$\underline{d} = \hat{\underline{r}} \times \underline{u}$ $\quad |\underline{d}| = |\underline{u}|$

$\underline{d} \perp \underline{u}_\perp \Rightarrow$ basis in $r_\perp$

$\Rightarrow \underline{v}_\perp = \cos\theta\, \underline{u}_\perp + \sin\theta\, \underline{d}$

Rotated vector:

$\underline{v} = \underline{v}_\parallel + \underline{v}_\perp = (\hat{\underline{r}} \otimes \hat{\underline{r}})\underline{u} + \cos\theta(\underline{\underline{I}} - \hat{\underline{r}} \otimes \hat{\underline{r}})\underline{u} + \sin\theta\, \hat{\underline{r}} \times \underline{u}$

Can we write: $\underline{v} = \underline{\underline{Q}}(\hat{\underline{r}}, \theta)\, \underline{u}$?

Axial Tensor

Need to write $\underline{r} \times \underline{u} = \underline{\underline{R}}\,\underline{u}$!

$\underline{\underline{R}}\,\underline{u} = R_{ij} u_j \underline{e}_i$ and $\underline{r} \times \underline{u} = \epsilon_{mnl} r_m u_n \underline{e}_l$

$R_{ij} u_j \underline{e}_i = \epsilon_{mnl} r_m u_n \underline{e}_l$!

all indices are dummies $\Rightarrow$ rename

$l \to i$: $R_{ij} u_j \underline{e}_i = \epsilon_{mni} r_m u_n \underline{e}_i$

$R_{ij} u_j = \epsilon_{mni} r_m u_n$ $\quad i$ = free index

$n \to j$: $R_{ij} u_j = \epsilon_{mji} r_m u_j$

$R_{ij} = \epsilon_{mji} r_m$ $\quad i, j$ = free $\quad m$ = dummy

$m \to k$: $R_{ij} = \epsilon_{kji} r_k$

prop. of ϵ: $\epsilon_{kji} = -\epsilon_{jki} = \epsilon_{ikj}$

$$\boxed{R_{ij} = \epsilon_{ikj} r_k}$$

$tr(\underline{\underline{R}}) = 0$

$\underline{\underline{R}} = -\underline{\underline{R}}^T$ skew sym.

$$\underline{\underline{R}} = R_{ij} \underline{e}_i \otimes \underline{e}_j = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix}$$

$R_{12} = \epsilon_{132} r_3 = -r_3$ $\quad R_{13} = \epsilon_{123} r_2 = r_2$ $\quad R_{23} = \epsilon_{213} = -r_1$

Back to rotation

$$\underline{v} = (\underline{r}\otimes\underline{r})\underline{u} + \cos\theta\,(\underline{\underline{I}} - \underline{r}\otimes\underline{r})\underline{u} + \sin\theta\,\underline{\underline{R}}\,\underline{u}$$

$$= \underbrace{\left[\underline{r}\otimes\underline{r} + \cos\theta\,(\underline{\underline{I}} - \underline{r}\otimes\underline{r}) + \sin\theta\,\underline{\underline{R}}\right]}_{\underline{\underline{Q}}(\underline{r},\theta)}\underline{u}$$

Euler representation of finite rotation tensor

$$\boxed{\begin{aligned}\underline{\underline{Q}}(\underline{r},\theta) &= \underline{r}\otimes\underline{r} + \cos\theta\,(\underline{\underline{I}} - \underline{r}\otimes\underline{r}) + \sin\theta\,\underline{\underline{R}}\\ Q_{ij}(\underline{r},\theta) &= r_i r_j + \cos\theta\,(\delta_{ij} - r_i r_j) + \sin\theta\,\epsilon_{ikj} r_k\end{aligned}}$$

Example: Rotation tensor around $\underline{e}_3$

$$\underline{\underline{Q}}(\underline{e}_3,\theta) = \underline{e}_3\otimes\underline{e}_3 + \cos\theta\,(\underline{\underline{I}} - \underline{e}_3\otimes\underline{e}_3) + \sin\theta\,\underline{\underline{R}}_3$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \cos\theta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\underline{Q}}(\underline{e}_3,\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotate $\underline{e}_1$ by 90° $(\frac{\pi}{2})$ counter clockwise

$\cos\left(\frac{\pi}{2}\right) = 0 \qquad \sin\left(\frac{\pi}{2}\right) = 1$

$$\underline{\underline{Q}}\left(\underline{e}_3, \frac{\pi}{2}\right) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\underline{Q}}\left(\underline{e}_3, \frac{\pi}{2}\right) \underline{e}_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \underline{e}_2 \quad \checkmark$$

Infinitesimal Rotations

$$\lim_{\theta \to 0} Q(\underline{\hat{r}}, \theta) = (\hat{r} \otimes \hat{r}) + \cancel{\cos\theta}^{\,1} (\underline{\underline{I}} - \underline{\hat{r}} \otimes \underline{\hat{r}}) + \cancel{\sin\theta}^{\,\theta} \underline{\underline{R}}$$

$$= \underline{\underline{I}} + \theta \underline{\underline{R}}$$

$\Rightarrow$ Axial tensor $\underline{\underline{R}}$ give infinitesimal rotation

$$\underline{v} = (\underline{\underline{I}} + \theta \underline{\underline{R}}) \underline{u}$$

$$\underline{v} = \underline{u} + \theta (\underline{\hat{r}} \times \underline{u})$$

$\Rightarrow$ cross product gives infinitesimal rotation

Determine θ and $\underline{r}$ from $\underline{\underline{Q}}$:

1) Find rotation angle

$$\mathrm{tr}(\underline{\underline{Q}}) = Q_{ii} = r_i r_i + \cos\theta\,(\delta_{ii} - r_i r_i) + \sin\theta\,\epsilon_{ikj}\, r_k$$

$$r_i r_i = \underline{r}\cdot\underline{r} = 1 \qquad \text{unit vector}$$

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$$

$$\epsilon_{iki} = 0$$

$$\Rightarrow \quad \mathrm{tr}(\underline{\underline{Q}}) = 1 + 2\cos\theta \quad \Rightarrow \quad \boxed{\cos\theta = \frac{\mathrm{tr}(\underline{\underline{Q}}) - 1}{2}}$$

Example: $\underline{\underline{Q}}\left(\underline{e}_3, \frac{\pi}{2}\right) \qquad \mathrm{tr}(\underline{\underline{Q}}) = 1$

$$\cos\theta = 0 \quad \Rightarrow \quad \theta = \frac{\pi}{2}$$

2) Axis of rotation $\underline{r}$:

$$\underline{\underline{R}} = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix} \quad \text{is axial tensor}$$

we are given $\underline{\underline{Q}}$ not $\underline{\underline{R}}$

$\Rightarrow$ Extract $\underline{\underline{R}}$ from $\underline{\underline{Q}}$

$$\underline{\underline{Q}} = \mathrm{sym}(\underline{\underline{Q}}) + \mathrm{skew}(\underline{\underline{Q}})$$

$$\mathrm{sym}(\underline{\underline{Q}}) = \tfrac{1}{2}(\underline{\underline{Q}} + \underline{\underline{Q}}^T) = \underline{r} \otimes \underline{r} + \cos\theta\,(\underline{\underline{I}} - \underline{r} \otimes \underline{r})$$

$$\mathrm{skew}(\underline{\underline{Q}}) = \tfrac{1}{2}(\underline{\underline{Q}} - \underline{\underline{Q}}^T) = \sin\theta\ \underline{\underline{R}} = \sin\theta\, \epsilon_{ikj}\, r_k\, \underline{e}_i \otimes \underline{e}_j$$

$$\tfrac{1}{2}(\underline{\underline{Q}} - \underline{\underline{Q}}^T) = \tfrac{1}{2}(Q_{ij} - Q_{ji})\ \underline{e}_i \otimes \underline{e}_j$$

equate two expressions for components

$$\underbrace{\tfrac{1}{2}(Q_{ij} - Q_{ji})}_{\text{know}} = \sin\theta\, \epsilon_{ikj}\, \underset{\substack{\uparrow \\ \text{want}}}{r_k}$$

remove ϵ_{ikj} using $\epsilon\delta$ identities

$$\epsilon_{ilj}\tfrac{1}{2}(Q_{ij} - Q_{ji}) = \sin\theta\, \epsilon_{ilj}\,\epsilon_{ikj}\, r_k$$

$$= \sin\theta\, \epsilon_{lij}\,\epsilon_{kij}\, r_k$$

$$= \sin\theta\, 2\, r_l$$

$$\boxed{\epsilon_{lij}\,\epsilon_{kij} = 2\delta_{lk}}$$

$$\Rightarrow \boxed{r_l = \frac{\epsilon_{ilj}(Q_{ij} - Q_{ji})}{4\sin\theta} \qquad \underline{r} = \frac{1}{2\sin\theta}\begin{bmatrix} Q_{32} - Q_{23} \\ Q_{13} - Q_{31} \\ Q_{21} - Q_{12} \end{bmatrix}}$$

Example: $\underline{\underline{Q}}\left(\underline{e}_3, \frac{\pi}{2}\right) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\underline{r} = \frac{1}{2\,\cancelto{1}{\sin\left(\frac{\pi}{2}\right)}} \begin{bmatrix} 0-0 \\ 0-0 \\ 1+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \underline{e}_3$$