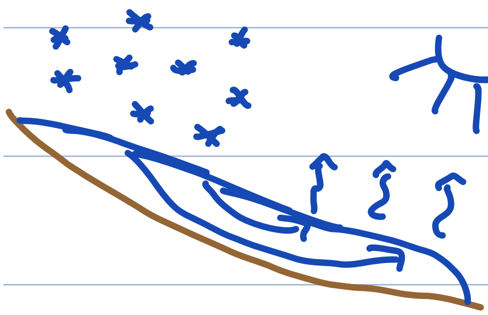
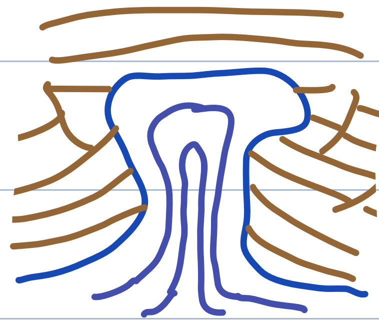
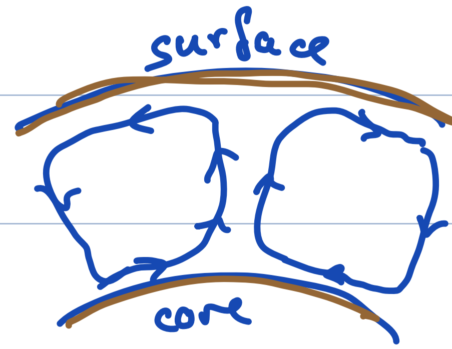
Viscous Flows

In Earth Science we are often interested in solids experience high temperature creep.

Glaciers | Salt domes | Mantle convection







$\Rightarrow$ modeled as very viscous fluids

Solid state creep is strongly affected by temperature and stress state.

$\Rightarrow$ viscosity is <u>not</u> constant $\quad \eta = \eta(T, \underline{v})$

Linear mom. balance:

$$\rho\left[\frac{\partial \underline{v}}{\partial t} + (\nabla \underline{v})\underline{v}\right] = \nabla \cdot \left[\underset{\substack{\uparrow \\ \text{not constant}}}{\eta}\left(\nabla \underline{v} + \nabla \underline{v}^T\right)\right] - \nabla \pi$$

Stokes Equation

dimensionless viscosity: $\eta' = \frac{\eta}{\eta_c}$ $\quad \eta_c$ = char. visc.

scaling to viscous term

$$\frac{\rho_0 v_c}{t_c}\frac{\partial \underline{v}'}{\partial t'} + \frac{\rho_0 v_c^2}{x_c}(\nabla' \underline{v}')\underline{v}' = \frac{\eta_c v_c}{x_c^2}\nabla' \cdot \left[\eta'(\nabla' \underline{v}' + \nabla' \underline{v}'^T)\right] - \frac{\pi_c}{x_c}\nabla' \pi'$$

divide by $\mu v_c / x_c^2$

$$\frac{x_c^2}{\nu t_c}\frac{\partial \underline{v}'}{\partial t'} + \frac{v_c x_c}{\nu}(\nabla' \underline{v}')\underline{v}' = \nabla' \cdot \left[\eta'(\nabla' \underline{v}' + \nabla' \underline{v}'^T)\right] - \underbrace{\frac{\pi_c x_c}{\mu v_c}}_{1}\nabla' \pi'$$

$$1 \Rightarrow \pi_c = \frac{\mu v_c}{x_c}$$

choose $t_c = t_A = \frac{x_c}{v_c}$

$$Re\left(\frac{\partial \underline{v}'}{\partial t'} + (\nabla' \underline{v}')\underline{v}'\right) = \nabla' \cdot \left[\eta'(\nabla' \underline{v}' + \nabla' \underline{v}'^T)\right] - \nabla' \pi'$$

In the limit $Re \ll 1$ we obtain

$$\Rightarrow \quad \nabla' \cdot \left[\eta'(\nabla' \underline{v}' + \nabla' \underline{v}'^T)\right] = \nabla' \pi'$$

Redimensionalize: $\underline{v}' = \frac{\underline{v}}{v_c}$ $\quad \pi' = \frac{\pi}{\eta_c v_c / x_c}$ $\quad x' = \frac{x}{x_c}$ $\quad \eta' = \frac{\eta}{\eta_c}$

$$\frac{x_c^2}{\eta_c v_c}\nabla \cdot \left[\eta(\nabla \underline{v} + \nabla \underline{v}^T)\right] = \frac{x_c^2}{\eta_c v_c}\nabla \pi$$

$$\Rightarrow \quad \nabla \cdot \left[\eta(\nabla \underline{v} + \nabla \underline{v}^T)\right] = \nabla \pi$$

Stokes equations with variable viscosity

$$\boxed{\begin{aligned} \nabla \cdot \left[\eta \left(\nabla \underline{v} + \nabla^T \underline{v} \right) \right] &= \nabla \pi \\ \nabla \cdot \underline{v} &= 0 \end{aligned}}$$

If $\eta = \text{const.}$ we have

$$\boxed{\begin{aligned} \eta \nabla^2 \underline{v} &= \nabla \pi \\ \nabla \cdot \underline{v} &= 0 \end{aligned}}$$

Properties of the Stokes Equation

1) Linearity

Construct solutions by linear superposition

2) Instanteneity

No time dependence other than due to time varying boundary conditions

3) Reversibility

If the body force and the velocity on boundary are reversed so is the velocity everywhere.

These tell us a lot about possible solutions.

Example: Sphere falling next to a wall