Why do we need tensors?

Scalars:

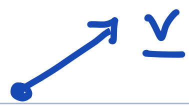
describe a quantity at a point

e.g. Temperature

Vectors:

describe quantity and a direction

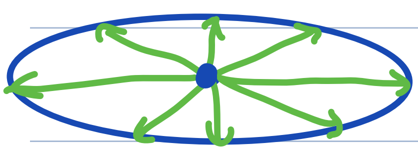
e.g. velocity (speed + direction)



Tensors:

describes how a quantity changes with direction

Think of an ellipsoid



Examples: anisotropic properties

stress, strain

moment of inertia

Second-order tensors

Linear operators : $\underline{v} = \underline{\underline{A}}\,\underline{u}$

maps vector $\underline{u} \in \mathcal{V}$ into vector $\underline{v} \in \mathcal{V}$

$\underline{\underline{A}}$ and $\underline{\underline{B}}$ are equal: $\underline{\underline{A}}\,\underline{v} = \underline{\underline{B}}\,\underline{v}$ for all $\underline{v} \in \mathcal{V}$

Special tensors

Zero tensor: $\underline{\underline{0}}\,\underline{v} = \underline{0}$ for all $\underline{v} \in \mathcal{V}$

Identity tensor: $\underline{\underline{I}}\,\underline{v} = \underline{v}$ for all $\underline{v} \in \mathcal{V}$

Basic algebraic operations

α = scalar, $\underline{v}$ = vector, $\underline{\underline{A}}$ & $\underline{\underline{B}}$ 2nd-ord. tensors

1) $(\alpha \underline{\underline{A}})\,\underline{v} = \underline{\underline{A}}\,(\alpha \underline{v})$ scalar multiplication

2) $(\underline{\underline{A}} + \underline{\underline{B}})\,\underline{v} = \underline{\underline{A}}\,\underline{v} + \underline{\underline{B}}\,\underline{v}$ tensor sum

3) $(\underline{\underline{A}}\,\underline{\underline{B}})\,\underline{v} = \underline{\underline{A}}\,(\underline{\underline{B}}\,\underline{v})$ tensor product

4) (tensor scalar product $\rightarrow$ later)

1 + 2 $\Rightarrow$ imply linearity

1, 2, 3 produce other tensors

set $\mathcal{V}^2$ of second order tensors $\Rightarrow$ vector space

Q: What is a basis for $\mathcal{V}^2$?

Representation of a tensor

In a frame $\{\underline{e}_i\}$ a second order tensor $\underline{\underline{S}}$ is represented by nine numbers

$$\boxed{S_{ij} = \underline{e}_i \cdot \underline{\underline{S}}\,\underline{e}_j}$$

Matrix representation of tensor in $\{\underline{e}_i\}$

$$[\underline{\underline{S}}] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \in \mathbb{R}^3 \times \mathbb{R}^3$$

Consider $\underline{v} = \underline{\underline{S}}\,\underline{u}$ where $\underline{v} = v_k \underline{e}_k$, $\underline{u} = u_j \underline{e}_j$

$$v_k \underline{e}_k = \underline{\underline{S}}(u_j \underline{e}_j) = \underline{\underline{S}}\,\underline{e}_j\, u_j$$

multiply by $\underline{e}_i$ from left

$$v_k\, \underline{e}_i \cdot \underline{e}_k = \underline{e}_i \cdot \underline{\underline{S}}\,\underline{e}_j\, u_j$$

$$v_k\, \delta_{ik} = \underline{e}_i \cdot \underline{\underline{S}}\,\underline{e}_j\, u_j$$

$$v_i = (\underline{e}_i \cdot \underline{\underline{S}}\,\underline{e}_j)\, u_j$$

$$\boxed{v_i = S_{ij}\, u_j}$$

Example: Multiply tensor by vector

$$\underline{\underline{S}} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 4 & 2 \\ 3 & 2 & 6 \end{bmatrix} \qquad \underline{u} = \begin{bmatrix} 7 \\ 3 \\ 6 \end{bmatrix}$$

Dot product with rows of $\underline{\underline{S}}$:

index notation: $v_i = S_{ij} u_j = \sum_{j=1}^{3} S_{ij} u_j$

$i=1: \; v_1 = S_{11} u_1 + S_{12} u_2 + S_{13} u_3 = 1 \cdot 7 + 0 \cdot 3 + 3 \cdot 6 = 25$

$$\underline{\underline{S}} = \begin{bmatrix} - & \underline{s}_1 & - \\ - & \underline{s}_2 & - \\ - & \underline{s}_3 & - \end{bmatrix} \qquad \begin{aligned} v_i &= \underline{s}_i \cdot \underline{u} \\ \underline{v} = v_i \underline{e}_i &= (\underline{s}_i \cdot \underline{u}) \underline{e}_i \end{aligned}$$

Linear combination of columns of $\underline{\underline{S}}$:

$$\underline{v} = \underline{\underline{S}}\,\underline{u} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 4 & 2 \\ 3 & 2 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ 6 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} + 6 \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ 0 \\ 21 \end{bmatrix} + \begin{bmatrix} 0 \\ 12 \\ 6 \end{bmatrix} + \begin{bmatrix} 18 \\ 12 \\ 36 \end{bmatrix} = \begin{bmatrix} 25 \\ 24 \\ 63 \end{bmatrix}$$

Dyadic Product

The dyadic product of two vectors $\underline{a}$ and $\underline{b}$ is the 2^{nd}-order tensor $\underline{a} \otimes \underline{b}$ defined by

$$\boxed{(\underline{a} \otimes \underline{b})\,\underline{v} = (\underline{b} \cdot \underline{v})\,\underline{a}} \qquad \text{for all } \underline{v} \in \mathcal{V}$$

This has the form: $\underline{\underline{S}}\,\underline{v} = \alpha\,\underline{a}$

in components: $S_{ij} v_j = \alpha\, a_i$

$\alpha = \underline{b} \cdot \underline{v} = b_j v_j$

$S_{ij} = [\underline{a} \otimes \underline{b}]_{ij}$

$\Rightarrow [\underline{a} \otimes \underline{b}]_{ij} v_j = b_j v_j a_i$

$[\underline{a} \otimes \underline{b}]_{ij} v_j = (a_i b_j) v_j$

$$\Rightarrow \quad [\underline{a} \otimes \underline{b}]_{ij} = a_i b_j$$

So that

$$\boxed{[\underline{a} \otimes \underline{b}] = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} = \underline{a}\,\underline{b}^T}$$

Linearity of dyadic product:

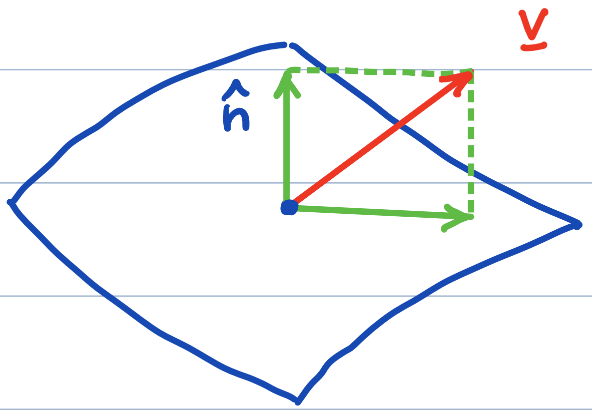
Linearity of dyadic product:

scalars α, β & vectors $\underline{a}, \underline{b}, \underline{v}, \underline{w}$

$$\boxed{(\underline{a} \otimes \underline{b})(\alpha \underline{v} + \beta \underline{w}) = \alpha(\underline{a} \otimes \underline{b})\underline{v} + \beta(\underline{a} \otimes \underline{b})\underline{w}}$$

Projection tensors

commonly used to partition forces on a surface.



$\underline{v} = \underline{v}_n^{\parallel} + \underline{v}_n^{\perp}$

Use dot product:

$\underline{v}_n^{\parallel} = (\underline{v} \cdot \hat{\underline{n}})\, \hat{\underline{n}}$

$\underline{v}_n^{\perp} = \underline{v} - \underline{v}_n^{\parallel}$

Tensors? $\underline{v}^{\parallel} = \underline{\underline{P}}^{\parallel} \underline{v}$, $\quad \underline{v}^{\perp} = \underline{\underline{P}}^{\perp} \underline{v}$

use dyadic property!

$\underline{v}_n^{\parallel} = (\underline{v} \cdot \hat{\underline{n}})\hat{\underline{n}} = (\hat{\underline{n}} \otimes \hat{\underline{n}})\, \underline{v} = \underline{\underline{P}}_n^{\parallel}\, \underline{v}$

$\underline{v}_n^{\perp} = \underline{v} - (\hat{\underline{n}} \otimes \hat{\underline{n}})\underline{v} = (\underline{\underline{I}} - \hat{\underline{n}} \otimes \hat{\underline{n}})\, \underline{v} = \underline{\underline{P}}_n^{\perp}\, \underline{v}$

Projection tensors:

$$\boxed{\underline{\underline{P}}_n^{\parallel} = \hat{\underline{n}} \otimes \hat{\underline{n}}} \qquad \boxed{\underline{\underline{P}}_n^{\perp} = \underline{\underline{I}} - \hat{\underline{n}} \otimes \hat{\underline{n}}}$$

Given any frame $\{\underline{e}_i\}$ the nine dyadic products $\{\underline{e}_i \otimes \underline{e}_j\}$ form a basis for V^2.
Any second-order tensor $\underline{\underline{S}}$ can be written as linear combination

Basis for $\mathcal{V}^2$

Given frame $\{\underline{e}_i\}$ the dyadic products $\{\underline{e}_i \otimes \underline{e}_j\}$ form a basis for vector space of tensors:

$$\boxed{\underline{\underline{S}} = S_{ij}\, \underline{e}_i \otimes \underline{e}_j} \quad \text{where} \quad S_{ij} = \underline{e}_i \cdot \underline{\underline{S}}\, \underline{e}_j$$

Consider $\underline{v} = \underline{\underline{S}}\,\underline{u}$ with $\underline{v} = v_i \underline{e}_i$, $\underline{u} = u_k \underline{e}_k$

$$\begin{aligned}
v_i \underline{e}_i &= S_{ij} (\underline{e}_i \otimes \underline{e}_j)(u_k \underline{e}_k) \\
&= S_{ij} u_k (\underline{e}_i \otimes \underline{e}_j) \underline{e}_k \quad \text{use dyadic property} \\
&= S_{ij} u_k (\underline{e}_j \cdot \underline{e}_k) \underline{e}_i \\
* \quad &= S_{ij} u_k \delta_{jk} \underline{e}_i \\
&= S_{ij} u_j \underline{e}_i
\end{aligned}$$

$$v_i \underline{e}_i = S_{ij} u_j \underline{e}_i$$

by comparison: $v_i = S_{ij} u_j$ as before

Note: Transfer property of δ_{ij} applies to tensors

$$\begin{aligned}
* \quad &= \underbrace{S_{ij}\delta_{jk}}_{S_{ik}} u_k \underline{e}_i \\
&= S_{ik} u_k \underline{e}_i
\end{aligned}$$

Dyadic property $\Rightarrow$ tensor vector products

What about tensor products?

$$\underline{\underline{S}}\,\underline{\underline{T}} = S_{ij}(\underline{e}_i \otimes \underline{e}_j)\, T_{kl}(\underline{e}_k \otimes \underline{e}_l) = S_{ij}\, T_{ij}(\underline{e}_i \otimes \underline{e}_j)(\underline{e}_k \otimes \underline{e}_l)$$

Product of dyadic products:

$$\boxed{(\underline{a} \otimes \underline{b})(\underline{c} \otimes \underline{d}) = (\underline{b} \cdot \underline{c})\, \underline{a} \otimes \underline{d}}$$ $\Rightarrow$ HW2

$$\underline{\underline{H}} = \underline{\underline{S}}\,\underline{\underline{T}} = S_{ij} T_{kl} (\underline{e}_i \otimes \underline{e}_j)(\underline{e}_k \otimes \underline{e}_l)$$

$$= S_{ij} T_{kl} \underbrace{(\underline{e}_j \cdot \underline{e}_k)}_{\delta_{jk}} \underline{e}_i \otimes \underline{e}_l$$

$$= S_{ik} T_{kl}\, \underline{e}_i \otimes \underline{e}_l$$ rename $l \leftrightarrow j$

$$H_{ij}\, \underline{e}_i \otimes \underline{e}_j = S_{ik} T_{kj}\, \underline{e}_i \otimes \underline{e}_j$$

$$\Rightarrow \quad \boxed{H_{ij} = S_{ik} T_{kj}}$$ note the dummy k !

Transpose of a Tensor

Every tensor $\underline{\underline{S}}$ has a transpose $\underline{\underline{S}}^T$ so that

$$\boxed{\underline{\underline{S}}\,\underline{u}\cdot\underline{v} = \underline{u}\cdot\underline{\underline{S}}^T\underline{v}} \quad \text{for all } \underline{u},\underline{v}\in\mathcal{V}$$

Implies $S^T_{ij} = S_{ji}$:

$$(S_{ij}u_j\underline{e}_i)\cdot(v_l\underline{e}_l) = (u_k\underline{e}_k)\cdot(S^T_{ij}v_j\underline{e}_i)$$

$$S_{ij}u_jv_l\underbrace{(\underline{e}_i\cdot\underline{e}_l)}_{\delta_{il}} = S^T_{ij}u_kv_j\underbrace{(\underline{e}_k\cdot\underline{e}_i)}_{\delta_{ki}}$$

$$S_{ij}u_jv_i = S^T_{ij}u_iv_j \quad \text{indices on } u \,\&\, v$$

$$S_{ji}u_iv_j = S^T_{ij}u_iv_j \quad i\leftrightarrow j \text{ on l.h.s}$$

$$\Rightarrow \quad \boxed{S^T_{ij} = S_{ji}} \quad \text{index notation for transpose}$$

Properties: 1) $(\underline{\underline{A}}^T)^T = \underline{\underline{A}}$

2) $(\underline{\underline{A}}\,\underline{\underline{B}})^T = \underline{\underline{B}}^T\underline{\underline{A}}^T$

3) $(\underline{u}\otimes\underline{v})^T = \underline{v}\otimes\underline{u}$

Symmetric & Skew Tensors

$\underline{\underline{S}}$ is symmetric if $\underline{\underline{S}} = \underline{\underline{S}}^T$ $\qquad S_{ij} = S_{ji}$

$\underline{\underline{S}}$ is skew if $\underline{\underline{S}} = -\underline{\underline{S}}^T$ $\qquad S_{ij} = -S_{ji}$

skew tensors related to rotation

Symmetric - Skew Decomposition

$$\boxed{\begin{aligned} \underline{\underline{S}} &= \underline{\underline{E}} + \underline{\underline{W}} \\ \underline{\underline{E}} &= \tfrac{1}{2}(\underline{\underline{S}} + \underline{\underline{S}}^T) \\ \underline{\underline{W}} &= \tfrac{1}{2}(\underline{\underline{S}} - \underline{\underline{S}}^T) \end{aligned}}$$

$$\underline{\underline{E}} = \underline{\underline{E}}^T$$

$$\underline{\underline{W}} = -\underline{\underline{W}}^T$$

Trace of Tensor

$$\boxed{tr(\underline{\underline{S}}) = S_{ii} = S_{11} + S_{22} + S_{33}}$$

Properties:

1) $tr(\underline{\underline{A}}^T) = tr(\underline{\underline{A}})$
2) $tr(\underline{\underline{A}}\,\underline{\underline{B}}) = t(\underline{\underline{B}}\,\underline{\underline{A}})$
3) $tr(\underline{\underline{A}} + \underline{\underline{A}}) = tr(\underline{\underline{A}}) + tr(\underline{\underline{B}})$
4) $tr(\alpha\,\underline{\underline{A}}) = \alpha\, tr(\underline{\underline{A}})$

Trace of dyadic product: $\boxed{tr(\underline{a} \otimes \underline{b}) = a_i b_i}$

$$tr(\underline{\underline{S}}) = tr(S_{ij}\, \underline{e}_i \otimes \underline{e}_j) \overset{\text{Prop. 4}}{=} S_{ij} \underbrace{tr(\underline{e}_i \otimes \underline{e}_j)}_{\delta_{ij}}$$
$$= S_{ij}\,\delta_{ij}$$
$$= S_{ii}$$

Spherical - Deviatoric Decomposition

$$\boxed{\underline{\underline{S}} = \alpha \underline{\underline{I}} + dev(\underline{\underline{S}})}$$

Spherical tensor: $sph(\underline{\underline{S}}) = \alpha \underline{\underline{I}}$ where $\alpha = \frac{1}{3} tr(\underline{\underline{S}})$

Deviatoric tensor: $dev(\underline{\underline{S}}) = \underline{\underline{S}} - \alpha \underline{\underline{I}}$

Important

Spherical part $\rightarrow$ volumetric stress/strain

Deviatoric part $\rightarrow$ shear stress/strain

Note: $tr(dev(\underline{\underline{S}})) = 0$ by design

Determinant and Inverse

$$\boxed{\det(\underline{\underline{A}}) = \det \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = \epsilon_{ijk} [\underline{\underline{A}}]_{i1} [\underline{\underline{A}}]_{j2} [\underline{\underline{A}}]_{k3}}$$

where $[\underline{\underline{A}}]_{i1}$, $[\underline{\underline{A}}]_{j2}$, $[\underline{\underline{A}}]_{k3}$ are the columns of $[\underline{\underline{A}}]$

Triple scalar product: $(\underline{a} \times \underline{b}) \cdot \underline{c} = \det[\underline{a}, \underline{b}, \underline{c}] = \epsilon_{ijk} a_i b_j c_k$

determinants $\Rightarrow$ volumes

Properties:
1) $\det(\underline{\underline{A}}\,\underline{\underline{B}}) = \det(\underline{\underline{A}}) \det(\underline{\underline{B}})$
2) $\det(\underline{\underline{A}}^T) = \det(\underline{\underline{A}})$
3) $\det(\alpha \underline{\underline{A}}) = \alpha^n \det(\underline{\underline{A}})$ ($\underline{\underline{A}}$ is $n \times n$)

$\underline{\underline{A}}$ is singular if $\det \underline{\underline{A}} = 0$.

If $\det \underline{\underline{A}} \neq 0$ then the inverse $\underline{\underline{A}}^{-1}$ exists

$$\boxed{\underline{\underline{A}}^{-1} \underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \underline{\underline{I}}}$$