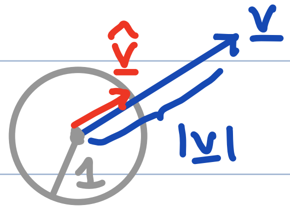
Vectors and index notation

Def: Vector $\underline{v}$ is a quantity with a magnitude and a direction

$\underline{v} = |\underline{v}|\, \hat{\underline{v}}$ $\qquad$ $|\underline{v}| \geq 0$ magnitude (scalar)

$\hat{\underline{v}} = \dfrac{\underline{v}}{|\underline{v}|}$ direction (unit vector)



Physical examples: velocity, force, heat flux

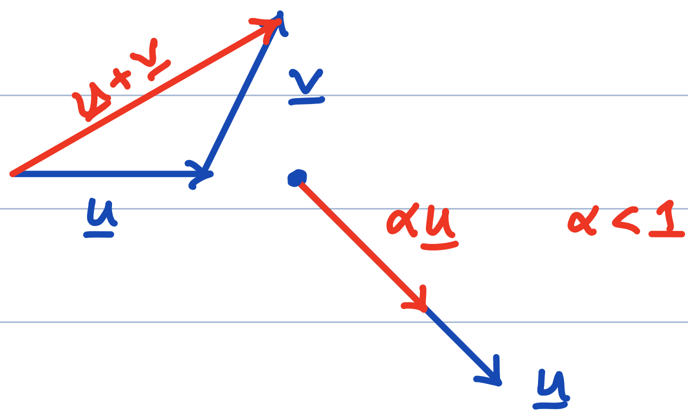
Q: Is it possible to have a vector with out direction?

Def: Vector space, $\mathcal{V}$, is a collection of objects that is closed under addition and scalar multiplication

$\underline{u} \in \mathcal{V}$ $\qquad$ $\underline{v} \in \mathcal{V}$ $\qquad$ $\alpha \in \mathbb{R}$

1) $\underline{u} + \underline{v} \in \mathcal{V}$

2) $\alpha \underline{u} \in \mathcal{V}$



Q: Do vectors in $\mathbb{R}^+$ form a vector space?

Basis for a vector space

Def.: Basis for $\mathcal{V}$ is a set of linearly independent vectors $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ that span the space (3D).

components of vector in basis

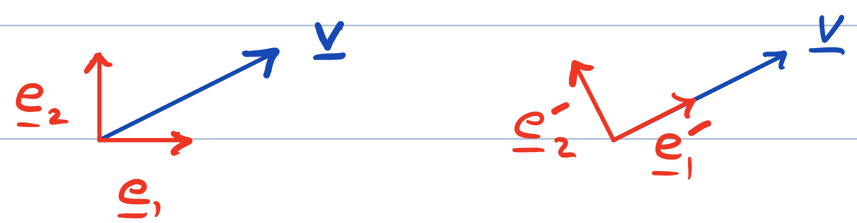
$$\underline{v} = v_1 \underline{e}_1 + v_2 \underline{e}_2 + v_3 \underline{e}_3 \qquad [\underline{v}] = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Here $[\underline{v}]$ is the representation of $\underline{v}$ in $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$

$$\boxed{\text{vector} \longleftrightarrow \text{representation}}$$

basis is not unique !

Example:



$$[\underline{v}] = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad [\underline{v}]' = \begin{bmatrix} \sqrt{5} \\ 0 \end{bmatrix}$$

same vector $\underline{v}$ two different representations !

Reference frame

any orthonormal basis $\{\underline{e}\} = \{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$

normal: $|\underline{e}_1| = |\underline{e}_2| = |\underline{e}_3| = 1$

ortho: $\underline{e}_1 \perp \underline{e}_2$, $\underline{e}_1 \perp \underline{e}_3$, $\underline{e}_2 \perp \underline{e}_3$

Q: Any additional common restrictions on basis?

Index notation

1) Dummy index

$\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$

$$\underline{v} = v_1 \underline{e}_1 + v_2 \underline{e}_2 + v_3 \underline{e}_3 = \sum_{i=1}^{3} v_i \underline{e}_i \equiv v_i \underline{e}_i$$

If index is repeated twice in a term

$\Rightarrow$ summation is implied

(Einstein summation convention)

$\Rightarrow$ dummy index

Note: $\underline{a} = a_i \underline{e}_i = a_k \underline{e}_k = a_q \underline{e}_q$

$\Rightarrow$ rename dummy indices

Example: Collecting terms

$\underline{a} = a_i \underline{e}_i \qquad \underline{b} = b_j \underline{e}_j$

$$\underline{a} + \underline{b} = a_i \underline{e}_i + b_j \underline{e}_j \qquad \text{rename } j \to i$$

$$= a_i \underline{e}_i + b_i \underline{e}_i$$

$$= (a_i + b_i)\, \underline{e}_i$$

2) Free index

occurs only once in a term

Example: $a_i = c_j b_j b_i$ j = dummy index
i = free index

$\Rightarrow$ set of equations $i, j \in \{1, 2, 3\}$

$a_1 = \left(\sum_{j=1} c_j b_j\right) b_1$, $a_2 = \left(\sum_{j=1} c_j b_j\right) b_2$, $a_3 = \left(\sum_{j=1} c_j b_j\right) b_3$

Basis: $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\} = \{\underline{e}_i\}$

Note: - all terms must have same free indices
- there can be more than one free index
- same symbol cannot be used for both free and dummy index

Q: Why are these expressions meaningless?

1) $a_i = b_j$

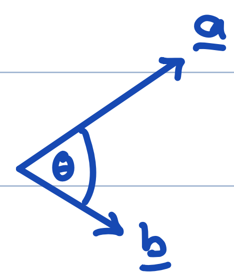
2) $a_i b_j = c_i d_j d_j$

3) $a_i b_j = c_i c_k d_k d_j + d_p c_e c_e d_q$

4) $a_i = b_k c_k d_k e_i$

Scalar product $\underline{a}, \underline{b} \in \mathcal{V}$

$$\boxed{\underline{a} \cdot \underline{b} = |\underline{a}|\,|\underline{b}| \cos\theta} \qquad \theta \in [0, \pi]$$



$\underline{a} \cdot \underline{b} = 0 \qquad \underline{a} \perp \underline{b}$

$\underline{a} \cdot \underline{a} = |\underline{a}|^2$

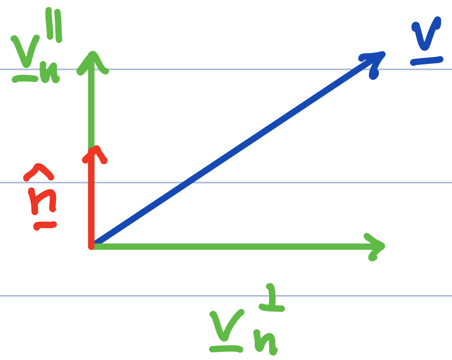
$\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$ commutative

Projection: $\hat{\underline{n}}$ = unit vector

$\underline{v} = \underline{v}_n^{\parallel} + \underline{v}_n^{\perp}$

$\underline{v}_n^{\parallel} = (\underline{v} \cdot \hat{\underline{n}})\, \hat{\underline{n}}$

$\underline{v}_n^{\perp} = \underline{v} - \underline{v}_n^{\parallel}$



$\Rightarrow$ components in a basis $\{\underline{e}_i\}$

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad \begin{aligned} v_1 &= \underline{v} \cdot \underline{e}_1 \\ v_2 &= \underline{v} \cdot \underline{e}_2 \\ v_3 &= \underline{v} \cdot \underline{e}_3 \end{aligned}$$

Kronecker Delta

To any frame $\{\underline{e}_i\}$ we associate

$$\boxed{\delta_{ij} = \underline{e}_i \bullet \underline{e}_j = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}}$$

$\Rightarrow$ orthonormal basis

δ_{ij} expresses scalar product in index notation

Properties: $\delta_{ij} = \delta_{ji}$ symmetry

$\underline{e}_i = \delta_{ij}\, \underline{e}_j$ transfer property

Examples:

Projection on basis

$\underline{u} = u_i \underline{e}_i$

$\underline{u} \bullet \underline{e}_j = (u_i \underline{e}_i) \bullet \underline{e}_j = u_i\, \underline{e}_i \bullet \underline{e}_j = u_i\, \delta_{ij} = u_j$

Scalar product: $\underline{a} = a_i \underline{e}_i \qquad \underline{b} = b_j \underline{e}_j$

$$\begin{aligned} \underline{a} \bullet \underline{b} &= (a_i \underline{e}_i) \bullet (b_j \underline{e}_j) = a_i b_j\, \underline{e}_i \bullet \underline{e}_j \\ &= a_i b_j\, \delta_{ij} \\ &= a_i b_i = a_j b_j \end{aligned}$$

Example of index manipulation:

$$(\underline{a} \cdot \underline{b})\, \underline{c} + \underline{b} = ?$$

$$\underline{a} = a_i \underline{e}_i \,, \quad \underline{b} = b_j \underline{e}_j \,, \quad \underline{c} = c_k \underline{e}_k$$

1) substitute:

$$(a_i \underline{e}_i \cdot b_j \underline{e}_j)\, c_k \underline{e}_k + b_j \underline{e}_j = ?$$

2) pull coefficients out to front

$$= a_i b_j c_k \,(\underline{e}_i \cdot \underline{e}_j)\, \underline{e}_k + b_j \underline{e}_j$$

3) Simplify using definition of dot product

$$= a_i b_j c_k \delta_{ij}\, \underline{e}_k + b_j \underline{e}_j$$

4) Use transfer property

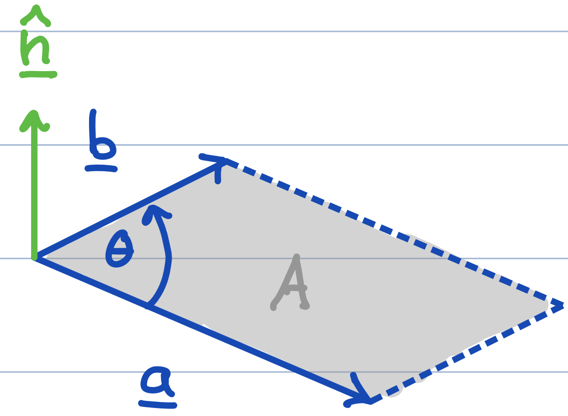
$$= a_i b_i c_k \underline{e}_k + b_j \underline{e}_j$$

5) Rename indicies to collect terms

$$= a_i b_i c_k \underline{e}_k + b_k \underline{e}_k$$

$$= (a_i b_i c_k + b_k)\, \underline{e}_k$$

Vector product



$\underline{a}, \underline{b} \in \mathcal{V}$

$$\boxed{\underline{a} \times \underline{b} = |\underline{a}|\,|\underline{b}| \sin\theta \, \hat{\underline{n}}}$$

$\theta \in [0, \pi]$

$\hat{\underline{n}}$ unit vector $\perp$ to $\underline{a}$ & $\underline{b}$

(right hand rule)

$|\underline{a} \times \underline{b}|$ = area of parallelogram

Q: Significance of $\underline{a} \times \underline{b} = \underline{0}$?

Note: $\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$ $\Rightarrow$ not commutative

Permutation symbol (Levi-Civita)

vector product in index notation

$$\boxed{\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \in \{123, 231, 312\} \text{ even perm.} \\ -1 & \text{if } ijk \in \{321, 213, 132\} \text{ odd perm.} \\ 0 & \text{repeated index} \end{cases}}$$

Flipping any 2 indices changes sign

$$\epsilon_{ijk} = -\epsilon_{kji} = -\epsilon_{jik} = -\epsilon_{ikj}$$

Invariant under cyclic permutation

$$\epsilon_{ijk} = \epsilon_{kij} = \epsilon_{jki}$$

Relation to frame $\{\underline{e}_i\}$

$$\boxed{\underline{e}_i \times \underline{e}_j = \epsilon_{ijk}\,\underline{e}_k}$$

Vector product: $\underline{a} \times \underline{b} = \underline{c}$

$\underline{a} = a_i\,\underline{e}_i \quad \underline{b} = b_j\,\underline{e}_j \quad \underline{c} = c_k\,\underline{e}_k$

$$\underline{a} \times \underline{b} = (a_i\underline{e}_i) \times (b_j\underline{e}_j) = a_i b_j\,(\underline{e}_i \times \underline{e}_j)$$

$$= \underbrace{a_i b_j\,\epsilon_{ijk}\,\underline{e}_k}_{=\,c_k\,\underline{e}_k} = \underline{c} =$$

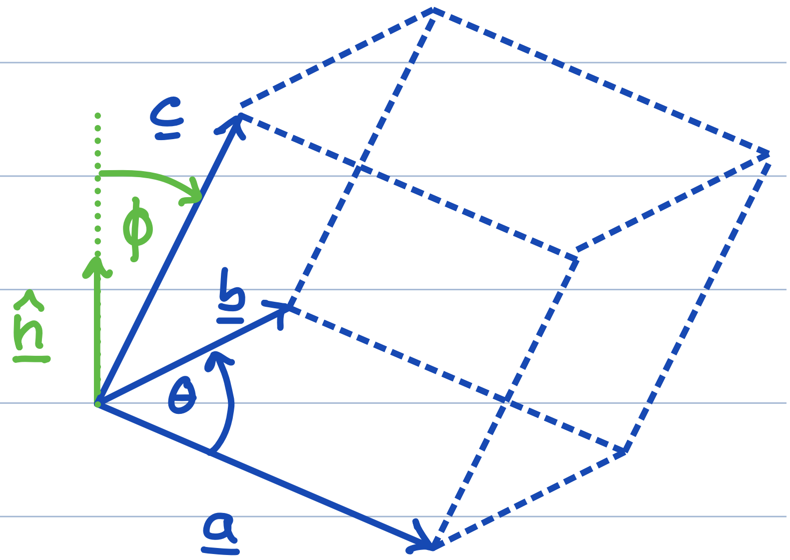
$$\Rightarrow \quad \boxed{c_k = \epsilon_{ijk}\,a_i b_j}$$

Triple scalar product

$\underline{a}, \underline{b}, \underline{c} \in \mathcal{V}$

$$\boxed{(\underline{a} \times \underline{b}) \cdot \underline{c} = |\underline{a}||\underline{b}||\underline{c}| \sin\theta \cos\phi}$$

$\Rightarrow$ Volume of the parallel epiped



$(\underline{a} \times \underline{b}) \cdot \underline{c} = \underline{0} \Rightarrow$ linearly dependent

$(\underline{a} \times \underline{b}) \cdot \underline{c} > 0 \Rightarrow$ right handed

$(\underline{a} \times \underline{b}) \cdot c < 0 \Rightarrow$ left handed

Cartesian reference frame

right handed orthonormal basis $\{\underline{e}_i\}$

$\Rightarrow \quad (\underline{e}_1 \times \underline{e}_2) \cdot \underline{e}_3 = 1$

Relation to Levi-Civita

$$\boxed{\epsilon_{ijk} = (\underline{e}_i \times \underline{e}_j) \cdot \underline{e}_k}$$

Proof: $\epsilon_{ijk} = (\underline{e}_i \times \underline{e}_j) \cdot \underline{e}_k$

$$= \epsilon_{ijl}\, \underline{e}_l \cdot \underline{e}_k$$

$$= \epsilon_{ijl}\, \delta_{lk} = \epsilon_{ijk} \checkmark$$

Use: $\underline{a} = a_i \underline{e}_i$, $\underline{b} = b_j \underline{e}_j$, $\underline{c} = c_k \underline{e}_k$

$$(\underline{a} \times \underline{b}) \cdot \underline{c} = \left((a_i \underline{e}_i) \times (b_j \underline{e}_j)\right) \cdot (c_k \underline{e}_k)$$

$$= a_i b_j c_k \underbrace{(\underline{e}_i \times \underline{e}_j) \cdot \underline{e}_k}_{\epsilon_{ijk}}$$

$$= \epsilon_{ijk}\, a_i b_j c_k$$

$$\boxed{(\underline{a} \times \underline{b}) \cdot \underline{c} = \epsilon_{ijk}\, a_i b_j c_k}$$

$\Rightarrow$ Invariant under cyclic permutation

$$(\underline{a} \times \underline{b}) \cdot \underline{c} = (\underline{c} \times \underline{a}) \cdot \underline{b} = (\underline{b} \times \underline{c}) \cdot \underline{a}$$

Relationship to determinant

matrix $[\underline{a}\ \underline{b}\ \underline{c}] = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$$\boxed{(\underline{a} \times \underline{b}) \cdot \underline{c} = \det([\underline{a}\ \underline{b}\ \underline{c}])}$$

determinants $\Rightarrow$ volumes

$$\underline{c} \cdot (\underline{a} \times \underline{b}) = \underline{a} \cdot (\underline{b} \times \underline{c})$$

$$\underline{b} \times \underline{c} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \underline{e}_1(b_2c_3 - b_3c_2) - \underline{e}_2(b_1c_3 - b_3c_1) + \underline{e}_3(b_1c_2 - b_2c_1)$$

taking dot product with $\underline{a}$ replaces first row

$$\Rightarrow \quad \underline{a} \cdot (\underline{b} \times \underline{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$