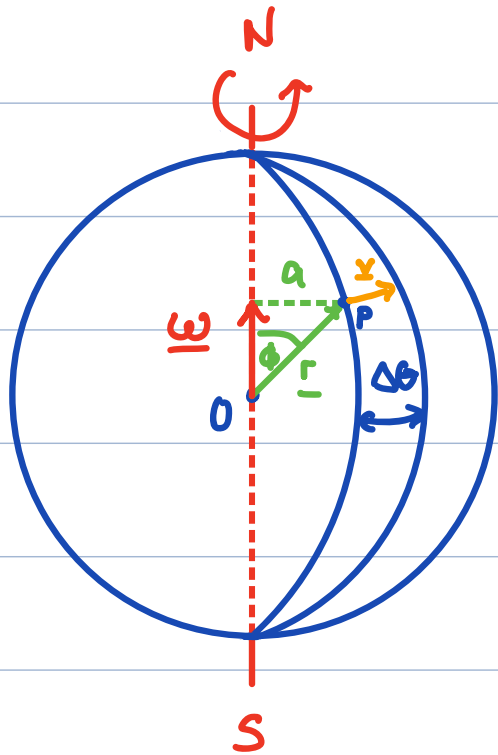


Angular Momentum and Torque

Rotational motion:



Angular velocity: $\underline{\omega} = |\underline{\omega}| \hat{\underline{e}}_{\omega}$

$$|\underline{\omega}| = \frac{d\theta}{dt}$$

Position vector: $\underline{r} = |\underline{r}| \hat{\underline{e}}_r$

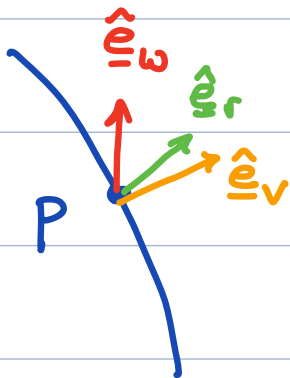
$$|\underline{r}| = \overline{OP}$$

Velocity: $\underline{v} = |\underline{v}| \hat{\underline{e}}_v$

$$|\underline{v}| = |\underline{\omega}| a = |\underline{\omega}| |\underline{r}| \sin \phi$$

$$\Rightarrow \underline{v} = |\underline{\omega}| |\underline{r}| \sin \phi \hat{\underline{e}}_v$$

local coord



$\hat{\underline{e}}_v \perp$ plane of $\hat{\underline{e}}_{\omega}$ and $\hat{\underline{e}}_r$

$$\hat{\underline{e}}_{\omega} \times \hat{\underline{e}}_r = \sin \phi \hat{\underline{e}}_v$$

substitute

$$\underline{v} = |\underline{\omega}| |\underline{r}| \hat{\underline{e}}_{\omega} \times \hat{\underline{e}}_r$$

$$\underline{v} = (|\underline{\omega}| \hat{\underline{e}}_{\omega}) \times (|\underline{r}| \hat{\underline{e}}_r)$$

Note

$|\underline{v}| > 0$:

$\Rightarrow$ counter clockwise

$$\underline{v} = \underline{\omega} \times \underline{r}$$

Example: Your current velocity in ATX (lat. $\sim 30^\circ$ N)

$$\phi \approx 60^\circ = \frac{\pi}{3} \approx 1.05 \quad |\underline{r}| = 6.37 \cdot 10^6 \text{ m}$$

$$|\underline{\omega}| = 7.3 \cdot 10^{-5} \frac{\text{rad}}{\text{sec}}$$

$$|\underline{v}| = |\underline{\omega}| |\underline{r}| \sin \phi \approx 403 \frac{\text{m}}{\text{s}}$$

$\Rightarrow$ GEO 320S Intro to Atmospheric Science

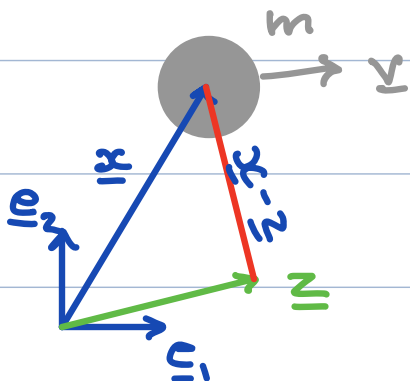
Angular momentum & torque

Linear momentum:

$$\underline{L} = m \underline{v}$$

angular momentum:

$$\underline{j} = (\underline{x} - \underline{z}) \times \underline{L}$$



relative to $\underline{z}$

force $\rightarrow$ linear momentum

torque $\rightarrow$ angular momentum

$$\text{torque: } \underline{\tau} = \frac{d\underline{j}}{dt} = \frac{d}{dt} [(\underline{x} - \underline{z}) \times m \underline{v}]$$

$$= m \frac{d}{dt} (\underline{x} \times \underline{v} - \underline{z} \times \underline{v})$$

$$\underline{\dot{x}} = \underline{v} \quad = m (\underline{\dot{x}} \times \underline{v} + \underline{x} \times \underline{\dot{v}} - \underline{\dot{z}} \times \underline{v} - \underline{z} \times \underline{\dot{v}})$$

$$\underline{a} = \underline{\dot{v}} \quad = m (\underline{v} \times \underline{v} + \underline{x} \times \underline{a} - \underline{z} \times \underline{a})$$

$$\underline{\tau} = (\underline{x} - \underline{z}) \times m \underline{a}$$

$$\underline{\tau} = (\underline{x} - \underline{z}) \times \underline{f}$$

units of torque. $[FL = \frac{ML^2}{T^2}]$ or Nm

torque = moment of force or moment

Two basic relations:

1) ang. & lin. mom.:

$$\underline{j} = (\underline{x} - \underline{z}) \times \underline{L}$$

2) torque & force:

$$\underline{\tau} = (\underline{x} - \underline{z}) \times \underline{f}$$

Resultant torque due to

1) body force:

$$\underline{\tau}_b = \int_V (\underline{x} - \underline{z}) \times \underline{b} \, dV$$

2) surface force:

$$\underline{\tau}_s = \int_{\Gamma} (\underline{x} - \underline{z}) \times \underline{t}_n \, dA$$

In rotation we have two important geometric locations in a body:

Center of volume:

$$\underline{x}_v = \frac{1}{V_B} \int_B \underline{x} \, dV$$

Center of mass:

$$\underline{x}_m = \frac{1}{m_B} \int_B \rho(\underline{x}) \underline{x} \, dV$$

Note: $\rho = \text{const}$

$$\underline{x}_m = \frac{1}{m_\Omega} \int_\Omega \rho \underline{x} \, dV = \frac{\rho}{\rho V_\Omega} \int_\Omega \underline{x} \, dV = \frac{1}{V_\Omega} \int_\Omega \underline{x} \, dV = \underline{x}_v$$

Gravitational body force

resultant around center of mass $\underline{x}_m = \text{const.}$

$$\underline{\tau}_b = \int_B (\underline{x} - \underline{x}_m) \times \rho \underline{g} dV \quad \underline{g} = \text{const.}$$

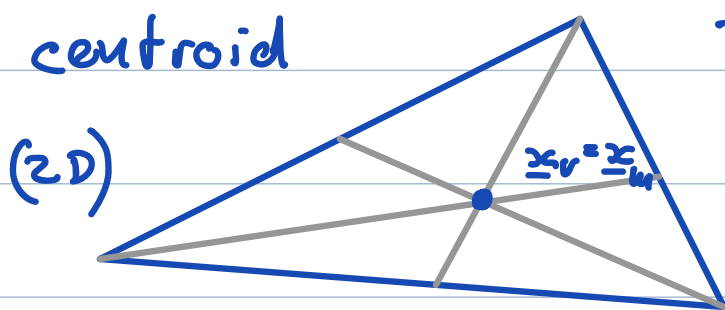
$$= \int_B \underline{x} \times \rho \underline{g} - \underline{x}_m \times \rho \underline{g} dV$$

$$= \underbrace{\int_B \underline{x} \rho dV}_m \times \underline{g} - \underline{x}_m \times \underline{g} \underbrace{\int_B \rho dV}_m$$

$$= m \underline{x}_m \times \underline{g} - \underline{x}_m \times m \underline{g}$$

$$= m (\underline{x}_m \times \underline{g} - \underline{x}_m \times \underline{g}) = \underline{0}$$

$\Rightarrow$ gravitational torque around center of mass vanishes



Triangle can be suspended from its centroid in any orientation without inducing rotation.

Moment of gravity:

torque due to gravity around origin $\underline{z}=0$

$$\underline{\tau}_G = \int_B \underline{r} \times \rho_b \underline{g} dV$$

Simplify "moment of gravity"

$$\underline{\tau}_G = \int_B \underline{r} \times \rho_b \underline{g} dV = \int_B (\underline{r} - \underline{r}_m + \underline{r}_m) \times \rho_b \underline{g} dV$$

$$= \int_B (\underline{r} - \underline{r}_m) \times \rho_b \underline{g} dV + \int_B \underline{r}_m \times \rho_b \underline{g} dV$$

$$= \underline{r}_m \times \underline{g} \underbrace{\int_B \rho_b dV}_m = \underline{r}_m \times m_b \underline{g}$$

$$\underline{\tau}_G = \underline{r}_m \times m_b \underline{g} = \underline{r}_m \times \underline{f}_G \quad \text{Moment of Gravity}$$

$\underline{f}_G = m_b \underline{g}$ = Weight (resultant force due to gravity)

center of mass $\Rightarrow$ center of gravity

$$\underline{r}_m \equiv \underline{r}_G$$

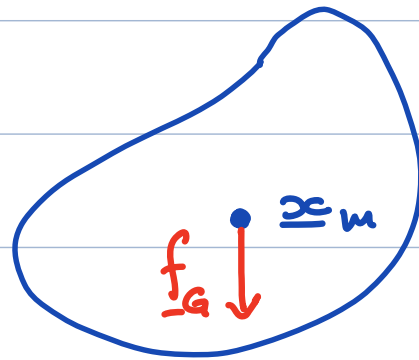
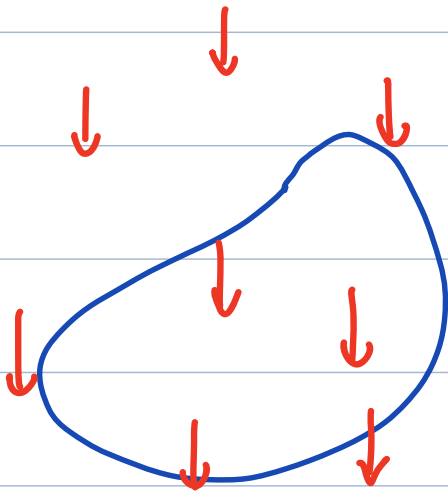
⇒ Gravity acts on the center of mass
(Center of mass theorem)

Continuum



Discrete

$$\underline{b}_G(\underline{x}) = \rho_b(\underline{x}) \underline{g}$$



force field can be represented as acting on a point where it does not induce a torque.

Hydrostatic surface force

Moment of buoyancy

Torque due to hydrostatic surface force around origin

$$\underline{\tau}_B = \oint_{\partial B} \underline{x} \times (-p \hat{n}) dA$$

$$\underline{t}_n = -p \hat{n}$$

Show that buoyancy acts on center of volume of displaced fluid

$$\underline{x}_B = \frac{1}{V} \int_B \underline{x} dV$$

$$\underline{\tau}_B = \oint_{\partial B} \underline{x}_v \times (-p \hat{n}) dA = 0$$

(requires some vector calculus $\Rightarrow$ later)

Use to simplify the moment of buoyancy

$$\Rightarrow \underline{\tau}_B = -\underline{x}_v \times (m_f g) = \underline{x} \times \underline{f}_B \quad \Rightarrow \text{HW2}$$

$$\underline{f}_B = -m_f g \quad (\text{resultant force due to buoyancy})$$

assumes body is fully submerged and $p_f = \text{const.}$

Center of volume $\Rightarrow$ center of buoyancy

$$\underline{x}_v \equiv \underline{x}_B$$

Hydrostatic moment

force balance: $\underline{f}_H = \underline{f}_G + \underline{f}_B = (m_b - m_f) \underline{g}$

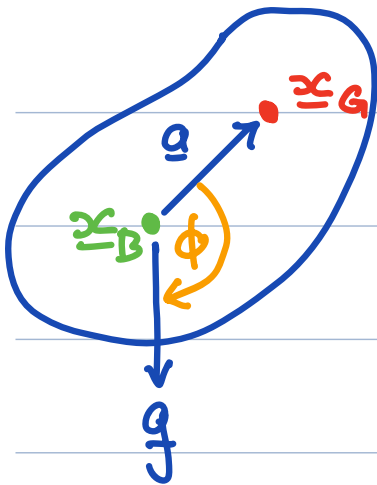
similarly

Torque balance: $\underline{\tau}_H = \underline{\tau}_G + \underline{\tau}_B$

$$= \underline{x}_G \times m_b \underline{g} - \underline{x}_B \times m_f \underline{g}$$

Neutrally buoyant / floating: $\underline{f}_H = \underline{0} \Rightarrow m_b = m_f = m$

Hydrostatic moment: $\underline{\tau}_H = m (\underline{x}_G - \underline{x}_B) \times \underline{g}$



sense of rotation of $\underline{\tau}_H$

$$\underline{\tau}_H = m \underline{a} \times \underline{g} \quad \underline{a} = \underline{x}_G - \underline{x}_B$$

$$= m |\underline{a}| |\underline{g}| \sin \phi \underline{\hat{n}},$$

• $\phi \in [0, \pi]$ smaller angle

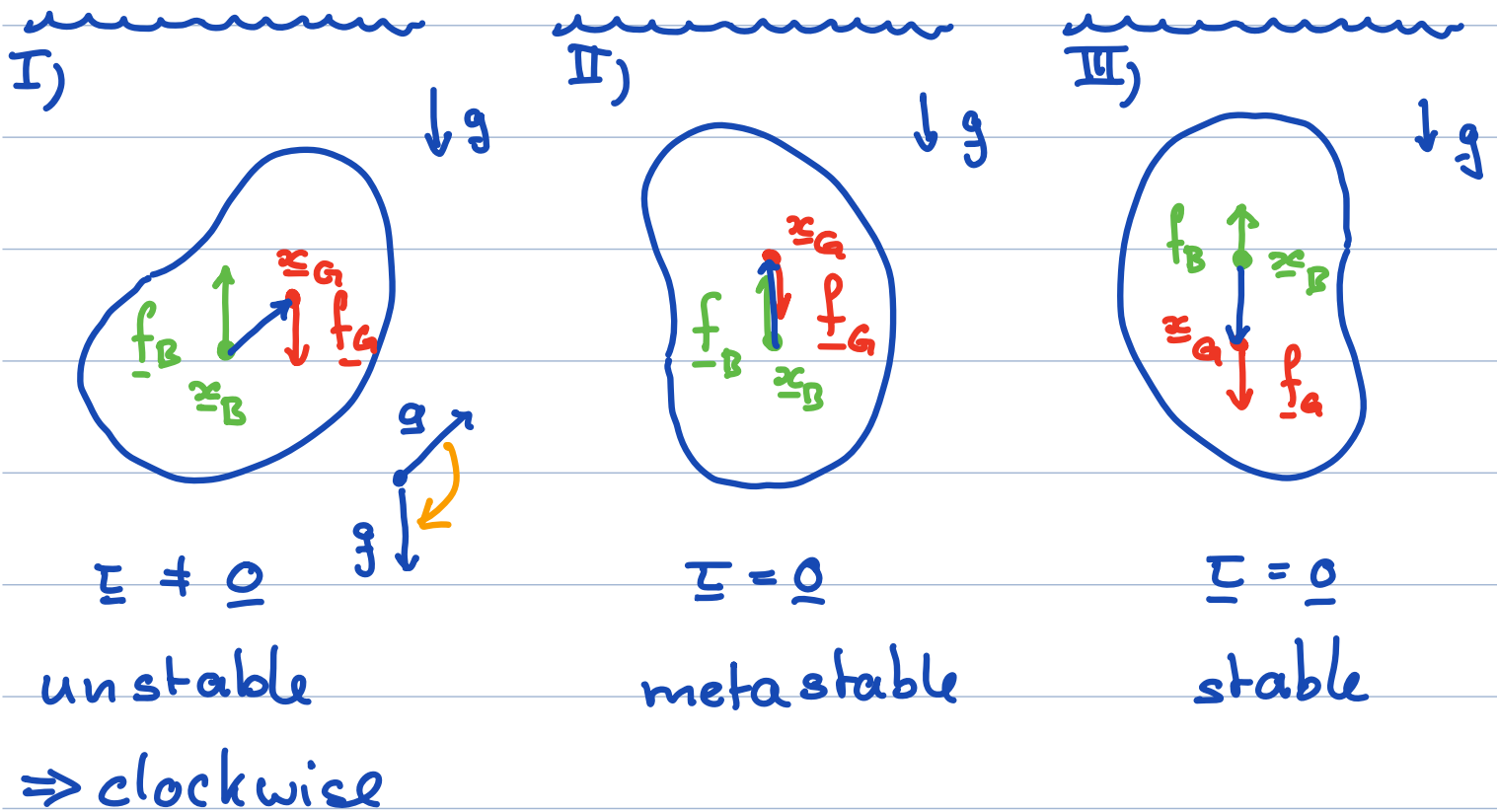
• $\underline{\hat{n}}$ direction - right hand rule

1) Counterclockwise relative to $\underline{\hat{n}}$

$\underline{\hat{n}}$ into board $\rightarrow$ clockwise

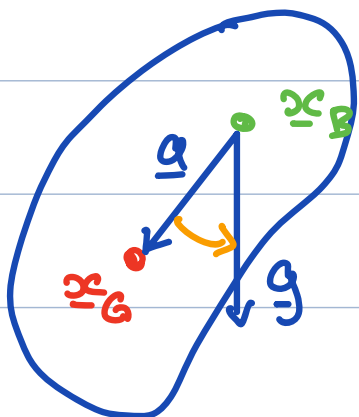
2) Sweeping from $\underline{a}$ to $\underline{g}$ along smaller angle

Stability of fully submerged body



II, small clock wise rotation $\rightarrow$ case I
 $\Rightarrow$ induces further clockwise rotation
 $\Rightarrow$ unstable

III, small clock wise rotation



$\Rightarrow$ induces counter clock wise rotation
 $\Rightarrow$ stable